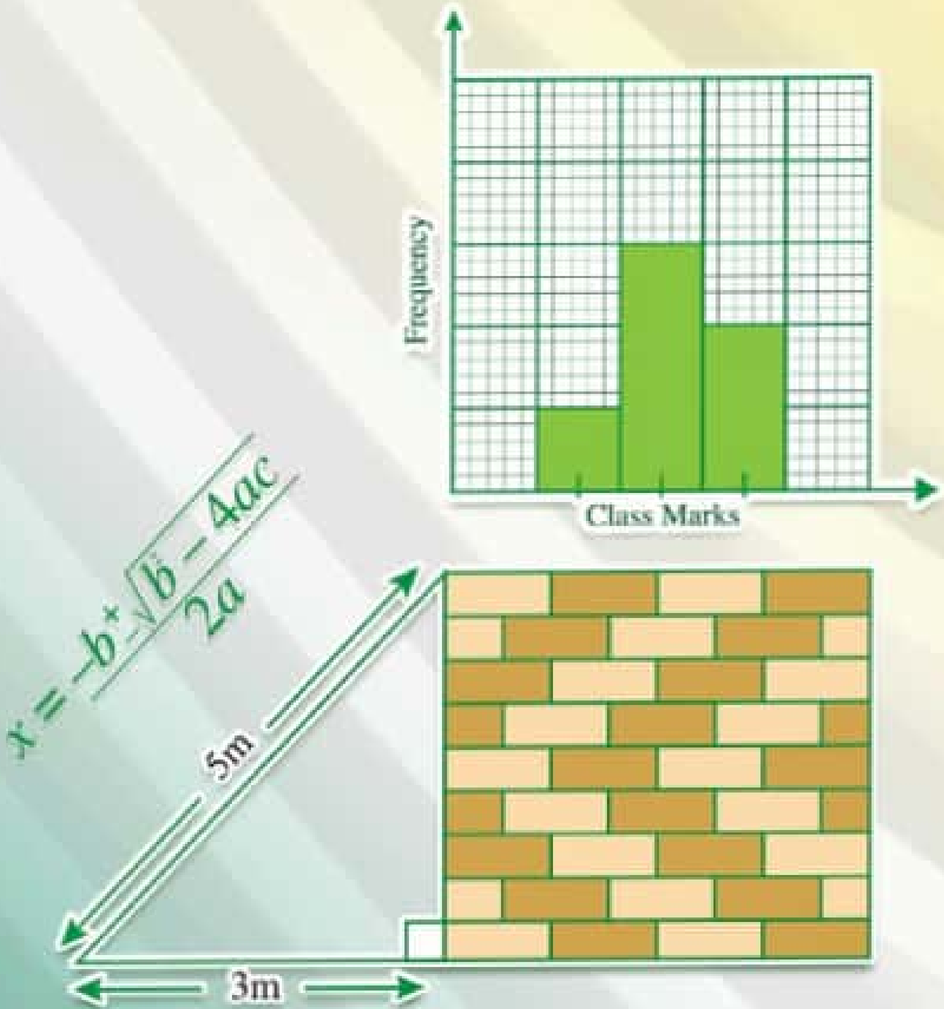
Basic Mathematics

for Secondary Schools

Student's Book

Form Two



Tanzania Institute of Education

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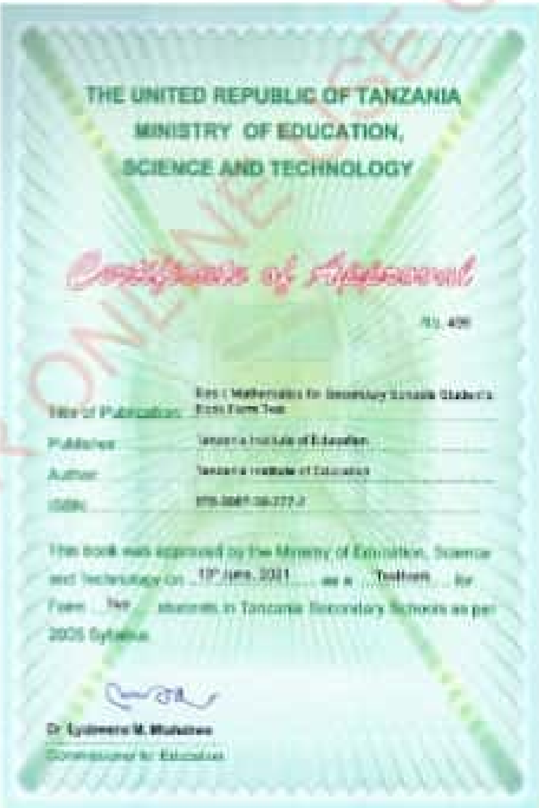
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Table of Contents

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Dr Aneth A. Komba
Director General
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Preface

This book, *Basic Mathematics Form Two*, is written specifically for a Form Two student in the United Republic of Tanzania. The book is prepared in accordance with the 2005 Basic Mathematics Syllabus for Secondary Schools Form I – IV, issued by the then Ministry of Education and Vocational Training.

The book consists of eleven chapters, namely Exponents and radicals; Algebra; Quadratic equations; Logarithms; Congruence; Similarity; Geometrical transformations; Pythagoras' theorem; Trigonometry; Sets; and Statistics. Each chapter contains activities, illustrations, and exercises. You are encouraged to do all the activities and exercises together with other assignments provided by your teacher. Doing so, will promote the development of the intended competencies.

Tanzania Institute of Education

Chapter One

Exponents and radicals

Introduction

Numbers can be expressed in different forms including exponential and radical forms. The exponent can be an integer or a fraction. An exponential expression with a fractional exponent can be expressed in another form called radical or root form. There is a close relationship between exponents and radicals. In this chapter, you will learn to write numbers in exponential form, to find square roots and cube roots of numbers, to rationalize denominators and to express formulae in different ways. The competences developed in this chapter will enable you to express numbers in short form and apply the knowledge to describe scientific phenomena such as population growth, magnitude of earthquakes, rates of spread of diseases and rate of growth or decay of bacteria.

Exponents

A product of the same number appearing repeatedly in the multiplication expression can be written in exponential form. Observe the following examples:

(i) $2 \times 2 \times 2 \times 2$

2 is multiplied repeatedly 4 times. In short form, this expression is written as 2^4. This means that, $2 \times 2 \times 2 \times 2 = 2^4$.

So, 2^4 is the exponential form of $2 \times 2 \times 2 \times 2$.

Likewise, $2 \times 2 \times 2 \times 2$ is the expanded form of 2^4.

(ii) $8 \times 8 \times 8 \times 8 \times 8 \times 8$

8 is multiplied repeatedly 6 times.

In short form, $8 \times 8 \times 8 \times 8 \times 8 \times 8$ is written as 8^6.

This means that, $8 \times 8 \times 8 \times 8 \times 8 \times 8 = 8^6$.

So, 8^6 is the exponential form of $8 \times 8 \times 8 \times 8 \times 8 \times 8$.

Likewise, $8 \times 8 \times 8 \times 8 \times 8 \times 8$ is the expanded form of 8^6.

Numbers in exponential form such as 2^4 and 8^6 are also called **exponential numbers**.

Read 2^4 as the "fourth power of 2" or "two raised to exponent four".
Read 8^6 as the "sixth power of 8" or "eight raised to exponent six".

In the number 2^4, 2 is the base and 4 is the exponent.
In the number 8^6, 8 is the base and 6 is the exponent.

From these two examples, we learn the following:

1. The numbers 2^4 and 8^6 are called **powers.**
2. The numbers 4 and 6 are **exponents.**

Activity 1.1: Identifying the power, base and exponent of an exponential expression.

Steps:

1. Form a group as instructed by the teacher.
2. Each member in the group has to choose any counting number and multiply it repeatedly n times, where n is any whole number less than 10.
3. Each member has to write the product in exponential form, identify the power, base and exponent.
4. Share your work in the group.
5. Thereafter, select one member in your group to present to the rest of the class. The presentation should show the number multiplied repeatedly n times, the product in exponential form, base, exponent and power.

Generally, a^n is a number written in exponential form, where a is a base and n is an exponent. The number a^n is called **power.**

There are two special exponents commonly used in mathematics.

These are **square units** (x^2) and **cubic units** (x^3). The square units are usually used to represent **area** and the cubic units are used to represent **volume.**

Example 1.1

Write each of the following expressions in exponential form:

(a) $2 \times 2 \times 2$

(b) $k \times k \times k \times k \times k$

(c) $(-3) \times (-3) \times (-3) \times (-3) \times (-3) \times (-3)$

(d) $m \times m \times m \times m$ ... *(n times)*

Solution

(a) 2^3 (b) k^5 (c) $(-3)^6$ (d) m^n

Example 1.2

Write each of the following expressions in power form:

(a) $6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6$

(b) $\left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right)$

(c) $10 \times 10 \times 10 \times 10 \times 10 \times 10$

(d) $(-7) \times (-7) \times (-7)$

Solution

(a) 6^8 (b) $\left(-\frac{3}{4}\right)^4$ (c) 10^6 (d) $(-7)^3$

Example 1.3

Give the power, base and exponent for each of the following exponential numbers:

(a) 4^3 (b) $(-10)^6$ (c) x^n

Solution

(a) 4^3 is the power, 4 is the base and 3 is the exponent.

(b) $(-10)^6$ is the power, -10 is the base and 6 is an exponent.

(c) x^n is the power, x is the base and n is an exponent.

Example 1.4

Find the value of each of the following exponential numbers:

(a) 3^4 (b) $(-5)^3$ (c) $(-7)^2$ (d) 2^6

Solution

(a) $3^4 = 3 \times 3 \times 3 \times 3 = 81$

(b) $(-5)^3 = -5 \times -5 \times -5 = 125$

(c) $(-7)^2 = -7 \times -7 = 49$

(d) $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$

Exercise 1.1

Answer the following questions:

1. Give the base and exponent for each of the following exponential numbers:

(a) 6^{17} (b) 7^4 (c) $(-10)^{19}$ (d) 3^{800}

(e) 6^2 (f) 50^0 (g) y^{25} (h) 17^6

(i) $\left(\frac{5}{6}\right)^9$ (j) $\left(\frac{1}{2}\right)^{17}$ (k) 19^{101} (l) $(-75)^8$

(m) $(x+y)^n$ (n) $(7+x)^8$

2. Express each of the following exponential numbers in expanded form:

(a) 7^2 (b) $(-3)^3$ (c) 10^3 (d) 2^6

(e) 9^1 (f) $(-y)^3$ (g) $(-99)^5$ (h) $\left(\frac{9}{10}\right)^5$

(i) $(-0.35)^4$ (j) $(0.67)^3$

3. Write each of the following expressions in exponential form and then give the base and the exponent:

(a) $7 \times 7 \times 7 \times 7$

(b) $(-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2)$

(c) $14 \times 14 \times 14$

(d) 19

(e) $(x+b)(x+b)(x+b)(x+b)$

(f) $(-r) \times (-r) \times (-r) \times (-r) \times (-r) \times (-r) \times (-r)$

(g) $50 \times 50 \times \ldots \times 50$ (30 times)

(h) $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$

(i) $(a+b)(a+b)$

(j) $\frac{w}{8} \times \frac{w}{8} \times \frac{w}{8} \times \frac{w}{8} \times \frac{w}{8} \times \frac{w}{8}$

(k) $0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.3$

(l) $v \times v \times v$

4. Find the value of each of the following exponential numbers:

(a) 5^2 (b) 7^2 (c) 20^3 (d) 12^2

(e) 1^3 (f) $(-9)^2$ (g) 30^3 (h) $(-2)^3$

(i) $(-3)^3$ (j) 2^5 (k) $(-5)^3$ (l) 10^3

5. Express each of the following numbers in exponential form using the given bases:

(a) 25 in base 5 (b) 36 in base 6

(c) 1 728 in base 12 (d) 16 in base 2

(e) 1 000 000 000 000 in base 10

6. Express each of the following numbers as powers of two different bases:

(a) 16 (b) 64 (c) 81 (d) 625 (e) 1 000

Laws of exponents

There are three groups of exponents which are based on positive integral exponents, negative integral exponents and zero exponents. Simplification of exponents with regard to these three categories is based on four laws of exponents which are; multiplication law, division law, power law, and zero power.

Multiplication law for exponents

Consider the product of two exponential numbers of the same base with positive integral exponents such as $5^3 \times 5^2$. The exponential numbers are written in expanded form as follows:

$5^2 = 5 \times 5$ and $5^3 = 5 \times 5 \times 5$

Therefore,

$$5^2 \times 5^3 = \underbrace{(5 \times 5)} \times \underbrace{(5 \times 5 \times 5)}$$

$$= \underbrace{5 \times 5 \times 5 \times 5 \times 5}$$

Then, find how many times 5 is multiplied by itself. The answer is 5 times, which gives the exponent. That is 5 multiplies by itself five times. The exponent 5 can also be obtained by adding the exponents of each base. That is, $2 + 3 = 5$.

Hence, $5^2 \times 5^3 = 5^{2+3} = 5^5$.

Similarly, if a is any number, express $a^4 \times a^2$ as

$$a^4 \times a^2 = (a \times a \times a \times a) \times (a \times a)$$
$$= a^{4+2}$$
$$= a^6$$

Therefore, $a^4 \times a^2 = a^6$

Generally, given any non-zero number x with positive exponents m and n, then, $x^m \times x^n = x^{m+n}$. This is called the multiplication law of exponents.

Example 1.5

Write each of the following products as a single exponential number:

(a) $7^6 \times 7^8$ (b) $6^4 \times 6^3 \times 6^4$

(c) $\left(\frac{5}{9}\right)^3 \times \left(\frac{5}{9}\right)^4$ (d) $(0.45)^2 \times (0.45)^4 \times (0.45)^{12}$

Solution

By the multiplication law of exponents:

(a) $7^6 \times 7^8 = 7^{6+8} = 7^{14}$ (b) $6^4 \times 6^3 \times 6^4 = 6^{4+3+4} = 6^{11}$

(c) $\left(\frac{5}{9}\right)^3 \times \left(\frac{5}{9}\right)^4 = \left(\frac{5}{9}\right)^{3+4} = \left(\frac{5}{9}\right)^7$

(d) $(0.45)^2 \times (0.45)^4 \times (0.45)^{12} = (0.45)^{2+4+12} = (0.45)^{18}$

Suppose the expression $(2^4)^3$ is to be written as a single exponent.

This number can be written in expanded form as follows:

$$(2^4)^3 = 2^4 \times 2^4 \times 2^4$$
$$= 2^{4+4+4}$$
$$= 2^{4\times3}$$
$$= 2^{12}$$

Therefore, $(2^4)^3 = 2^{4\times3} = 2^{12}$.

Similarly, given the number $(a^2)^4$ where a is any real number, the number can be written as a single exponent as follows:

$$\begin{aligned}(a^2)^4 &= a^2 \times a^2 \times a^2 \times a^2\\ &= a^{2+2+2+2}\\ &= a^{2\times 4}\\ &= a^8\end{aligned}$$

Therefore, $(a^2)^4 = a^{2\times 4} = a^8$.

Example 1.6

Write each of the following numbers as a single exponent:

(a) $((3)^6)^3$ (b) $\left(\left(\frac{3}{7}\right)^6\right)^2$ (c) $((0.7)^7)^3$

Solution

(a) $((3)^6)^3 = 3^{6\times 3} = 3^{18}$

(b) $\left(\left(\frac{3}{7}\right)^6\right)^2 = \left(\frac{3}{7}\right)^{6\times 2} = \left(\frac{3}{7}\right)^{12}$

(c) $((0.7)^7)^3 = (0.7)^{7\times 3} = 0.7^{21}$

The expression $(7 \times 5)^3$, can be written in expanded form as follows:

$$\begin{aligned}(7 \times 5)^3 &= (7 \times 5) \times (7 \times 5) \times (7 \times 5)\\ &= 7 \times 5 \times 7 \times 5 \times 7 \times 5\\ &= 7 \times 7 \times 7 \times 5 \times 5 \times 5\\ &= 7^3 \times 5^3\end{aligned}$$

Therefore, $(7 \times 5)^3 = 7^3 \times 5^3$.

Similarly, if a and b are real numbers, then $(a \times b)^4$ can be expanded as follows:

$$\begin{aligned}(a \times b)^4 &= (a \times b) \times (a \times b) \times (a \times b) \times (a \times b)\\ &= (a \times a \times a \times a) \times (b \times b \times b \times b)\\ &= a^4 \times b^4\end{aligned}$$

Therefore, $(a \times b)^4 = a^4 \times b^4$.

Generally, $(a \times b)^n = a^n \times b^n$, where a and b are real numbers.

Example 1.7

Express each of the following expressions in its simplest form.

(a) $(7^2 \times c^3)^4$ (b) $(4 \times b^2 \times c^3)^3$ (c) $((0.7)^4 \times (0.3)^2)^4$

Solution

(a) $(7^2 \times c^3)^4 = 7^{2\times 4} \times c^{3\times 4}$
$= 7^8 \times c^{12}$
$= 7^8 c^{12}$

(b) $(4 \times b^2 \times c^3)^3 = 4^{1\times 3} \times b^{2\times 3} \times c^{3\times 3}$
$= 4^3 \times b^6 \times c^9$
$= 4^3 b^6 c^9$

(c) $((0.7)^4 \times (0.3)^2)^4$
$((0.7)^4 \times (0.3)^2)^4 = (0.7)^{4\times 4} \times (0.3)^{2\times 4}$
$= (0.7)^{16} \times (0.3)^8.$

Example 1.8

Write each of the following expressions by grouping together the letters with the same exponents:

(a) d^4t^4 (b) $a^5 \times b^4$ (c) $3cd^2 \times 5c^5d^4$

Solution

(a) $d^4t^4 = (dt)^4$

(b) $a^5 \times b^4 = a \times a^4 \times b^4$
$= a \times (ab)^4$
$= a(ab)^4$

(c) $3cd^2 \times 5c^5d^4 = 3 \times 5 \times c \times c^5 \times d^2 \times d^4$
$= 15 \times c^6 d^6$
$= 15(cd)^6.$

Exercise 1.2

Answer the following questions:

1. Simplify each of the following expressions and give the answers as a single exponential number:

   (a) $10^2 \times 10^2$
   (b) $\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^5$
   (c) $r^4 \times r^8$
   (d) $\left(\frac{7}{6}\right)^2 \times \left(\frac{7}{6}\right)^{12}$
   (e) $3a^2b^3 \times 4a^4b^5$
   (f) $\left(\frac{17}{20}\right)^2 \times \left(\frac{17}{20}\right)^7 \times \left(\frac{17}{20}\right)$
   (g) $16^2 \times 16^4 \times 16^4$
   (h) $\left(\frac{3}{4}\right)^2 \times \left(\frac{3}{4}\right)^2$
   (i) $10^3 \times 10^0$
   (j) $(0.58)^4 \times (0.58)^4 \times (0.58)^{16}$
   (k) $3^{14} \times 3^{17} \times 3^1$
   (l) $10^p \times 10^r \times 10^{10}$

2. Write each of the following expressions as a single exponential number:

   (a) $(4^2)^4$
   (b) $(a^3)^2$
   (c) $(b^4)^3$
   (d) $(18^1)^{20}$
   (e) $(71^3)^2$
   (f) $(a^2)^5$
   (g) $(19^6)^1$
   (h) $(23^2)^n$
   (i) $(m^2)^{17}$
   (j) $(x^{12})^6$
   (k) $(2^3)^7$
   (l) $\left(\left(\frac{3}{4}\right)^4\right)^4$

3. Write each of the following expressions such that each base is raised to a single exponent:

   (a) $(4\times3)^2$
   (b) $(2a)^2$
   (c) $(3\times7)^3$
   (d) $(2x^2)^2$
   (e) $(2r^2)^3$
   (f) $(10p^2)^3$
   (g) $5(mn)^2$
   (h) $3(a^2b)^3$
   (i) $(2a^3b^4)^2$
   (j) $(7x)^2 \times (7x)^m \times (7x)^3$

4. Write each of the following expressions as a single exponential number:

(a) $4^2 \times 5^3$ (b) $a^{17} \times b^{17}$ (c) $(2a)^5 \times a^5$

(d) $12k^{23} \times t^{23}$ (e) $2a^5 \times b^7$ (f) $3^2 \times 3^4$

(g) $3^2 \times 2^4$ (h) $12^{20} \times 12^{21}$ (i) $4a^5 \times b^6$

Division law of exponents

Consider the following example of dividing exponential numbers in the same base such as $7^5 \div 7^3$. The solution can be obtained as follows:

$$7^5 \div 7^3 = \frac{7^5}{7^3}$$
$$= \frac{7 \times 7 \times 7 \times 7 \times 7}{7 \times 7 \times 7}$$
$$= 7 \times 7$$
$$= 7^2$$

In this example, the exponent 2 can also be obtained by subtracting the exponent of the denominator from the exponent of the numerator while retaining the base as follows:

$$\frac{7^5}{7^3} = 7^{5-3}$$
$$= 7^2$$

Similarly, $\frac{a^7}{a^4} = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a}$

$$= a^{7-4}$$
$$= a^3$$

Therefore, $\frac{a^7}{a^4} = a^{7-4} = a^3$.

Generally, when dividing exponential numbers of the same base, subtract the exponent of the divisor from the exponent of the dividend.

That is: $\frac{x^m}{x^n} = x^{m-n}$, where $x \neq 0$.

Example 1.9

Simplify the following expressions:

(a) $\frac{3^{20}}{3^{12}}$ (b) $\frac{a^{28}}{a^{17}}$ (c) $\frac{(0.47)^{10}}{(0.47)^{6}}$

Solution

(a) $\frac{3^{20}}{3^{12}} = 3^{20-12}$

$= 3^8$

(b) $\frac{a^{28}}{a^{17}} = a^{28-17}$

$= a^{11}$

(c) $\frac{(0.47)^{10}}{(0.47)^{6}} = (0.47)^{10-6}$

$= (0.47)^4$

Similarly, when exponential numbers of different bases with the same exponents are divided, the number can be expressed as a single exponent. If a and b are any real numbers such that $b \neq 0$, then:

$$\frac{a^4}{b^4} = \frac{a \times a \times a \times a}{b \times b \times b \times b}$$

$$= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$$

$$= \left(\frac{a}{b}\right)^4$$

Generally, $\frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n$ where x, y and n are real numbers and $y \neq 0$

Example 1.10

Write each of the following expressions in the form of $\left(\frac{x}{y}\right)^n$.

(a) $\frac{13^2}{9^2}$ (b) $\frac{(0.5)^4}{(0.3)^4}$ (c) $\frac{16}{3^2}$

Solution

(a) $\frac{13^2}{9^2}=\left(\frac{13}{9}\right)^2$ (b) $\frac{(0.5)^4}{(0.3)^4}=\left(\frac{0.5}{0.3}\right)^4$ (c) $\frac{16}{3^2}=\frac{4^2}{3^2}=\left(\frac{4}{3}\right)^2$

Exercise 1.3

Simplify each of the following expressions:

1. $\frac{x^5}{x^2}$
2. $\frac{10^7}{10^4}$
3. $\frac{2^{7000}}{2^{6998}}$
4. $\frac{4x^4}{x^2}$
5. $33^3 \div 33^2$
6. $\frac{625}{5^4}$
7. $\frac{x^4 \times x^2}{x^3}$
8. $a^4 \div \left(2a^2 + 4a^2\right)$
9. $\frac{\left(2^5\right)^2}{2^2}$
10. $\frac{3^6}{27}$
11. $\frac{10^7}{10^3}$
12. $\frac{a^5 b^3}{ab^2}$
13. $\frac{(mn)^4}{m^2n^3}$
14. $\frac{k^3m^4n^6}{km^2n^3}$
15. $\frac{20a^3b}{5ab}$
16. $\frac{18a^4}{3a^3}$
17. $\frac{(2x)^2}{x^5}$
18. $\frac{12a^3b}{3ab}$
19. $\frac{15a^3b^2c^2}{5a^3bc}$
20. $\frac{3habc}{9c}$
21. $\frac{48a^4b^5}{8a^2b^3}$
22. $\frac{1.44t^3}{3.6t}$
23. $\frac{4a^2}{0.5a}$
24. $6r^3 \div \frac{1}{4r}$

25. $(6y)^3 \div (2y)^2$

26. $\dfrac{(3mn)^3}{8(mn)^3}$

27. $\dfrac{\left(3^2 \times 2^{13} \times 3^{10} \times 25\right)^3}{\left(3^{16} \times 2^{16} \times 5^2\right)^2}$

28. $\dfrac{\left(3^2 \times a^3 \times b^4 \times c^2\right)^3}{\left(14 \times c^2 \times a^2 \times b^3\right)^2}$

29. $\dfrac{\left(17^3 \times (4x)^2\right)^3}{\left((2x)^2 \times 17^2\right)^3}$

30. $\left(\dfrac{4}{3}\right)^3 \div \left(\dfrac{3}{2}\right)^3$

Zero exponent

An expression like $\dfrac{7^3}{7^3}$ can be simplified by using the division law of exponents as follows:

$$\frac{7^3}{7^3} = \frac{7 \times 7 \times 7}{7 \times 7 \times 7} = 1 \qquad (1)$$

Alternatively, $\dfrac{7^3}{7^3} = 7^{3-3} = 7^0 = 1$ (2)

From equations (1) and (2),

$7^0 = 1$

Similarly, if $m \neq 0$, then $\dfrac{m^4}{m^4} = \dfrac{m \times m \times m \times m}{m \times m \times m \times m} = 1$ (3)

Therefore, $\dfrac{m^4}{m^4} = m^{4-4} = m^0$ (4)

Hence, $m^0 = 1$ by equations (3) and (4)

> In general, for any non–zero number x, $x^0 = 1$.
>
> Also, note that, $x^0 = 1$ but 0^0 is not defined.

Negative exponents

The expression $\frac{8^3}{8^5}$, can be simplified using the division law of exponents as follows:

$$\frac{8^3}{8^5} = \frac{8 \times 8 \times 8}{8 \times 8 \times 8 \times 8 \times 8}$$
$$= \frac{1}{8 \times 8}$$
$$= \frac{1}{8^2}$$

Therefore, $\frac{8^3}{8^5} = \frac{1}{8^2}$ (1)

Also, $\frac{8^3}{8^5} = 8^{3-5} = 8^{-2}$ (2)

By comparing equations (1) and (2) it follows that, $8^{-2} = \frac{1}{8^2}$

Similarly, $\frac{k^5}{k^8} = \frac{k \times k \times k \times k \times k}{k \times k \times k \times k \times k \times k \times k \times k}$

$$= \frac{1}{k \times k \times k}$$
$$= \frac{1}{k^3}$$

Therefore, $\frac{k^5}{k^8} = \frac{1}{k^3} = k^{-3}$

In general, if $x \neq 0$, $x^{-n} = \frac{1}{x^n}$. When $n = 1$, $x^{-n} = x^{-1} = \frac{1}{x^1} = \frac{1}{x}$.

If $x \neq 0$, $\frac{1}{x}$ is called the reciprocal of x, which can also be written as x^{-1}.

Example 1.11

Express the following exponential numbers using positive exponents:

(a) 4^{-3} (b) a^{-7} (c) $\frac{1}{19^{-2}}$

Solution

(a) $4^{-3}=\left(4^{-1}\right)^{3}$

$=\left(\frac{1}{4}\right)^{3}$

$=\frac{1}{4^{3}}$

(b) $a^{-7}=\left(a^{-1}\right)^{7}$

$=\left(\frac{1}{a^{1}}\right)^{7}$

$=\frac{1}{a^{7}}$

(c) $\left(\frac{1}{19^{-2}}\right)=\frac{1}{\left(\frac{1}{19}\right)^{2}}=\frac{1}{\frac{1}{19^{2}}}=19^{2}$

Example 1.12

Simply the following expressions and give your answer in positive exponents:

(a) $\frac{x^{5}}{x^{7}}$ (b) $\frac{a^{4}b^{17}}{a^{10}b^{11}}$

Solution

(a) $\frac{x^{5}}{x^{7}}=x^{5-7}$

$=x^{-2}$

$=\frac{1}{x^{2}}$

Therefore, $\frac{x^{5}}{x^{7}}=\frac{1}{x^{2}}$

(b) $\frac{a^{4}b^{17}}{a^{10}b^{11}}=\frac{a^{4}}{a^{10}}\times\frac{b^{17}}{b^{11}}$

$=a^{4-10}b^{17-11}$

$=a^{-6}b^{6}$

$=\left(\frac{b^{6}}{a^{6}}\right)$

$=\left(\frac{b}{a}\right)^{6}$

Therefore, $\frac{a^{4}b^{17}}{a^{10}b^{11}}=\left(\frac{b}{a}\right)^{6}$.

Example 1.13

Simplify the following expressions and give your answers in negative exponents:

(a) $\frac{3^9}{3^{16}}$ (b) $\frac{(0.7)^6}{(0.7)^2}$

Solution

(a) $\frac{3^9}{3^{16}} = 3^{9-16}$

$= 3^{-7}$

Therefore, $\frac{3^9}{3^{16}} = 3^{-7}$

(b) $\frac{(0.7)^6}{(0.7)^2} = (0.7)^{6-2}$

$= (0.7)^4$

$= \frac{1}{(0.7)^{-4}}$

Therefore, $\frac{(0.7)^6}{(0.7)^2} = \frac{1}{(0.7)^{-4}}$

Exercise 1.4

In questions 1 to 30 simplify the given expressions and give the answers in positive or zero exponents:

1. $\frac{x^4}{x^7}$
2. $\frac{x^6}{x^8}$
3. $\frac{x}{x^{11}}$
4. $\frac{y^4 \times y^7}{y^{15}}$
5. $\frac{r^5 \times r^8}{7^7 \times r^4}$
6. $\frac{a^{21} \times a^3 \times a^2}{a^{22} \times a^4 \times a}$
7. $\frac{3^2 \times 3^4 \times 3^6}{3^3 \times 3^3 \times 3^7}$
8. $\frac{a^5 \times 6^0}{a^6 \times a^6}$
9. $\frac{a^7 \times 6^7}{a^9 \times a^7}$
10. $\frac{2^5 \times 5^2}{2^{10} \times 10^4}$
11. $\frac{2^3 \times 4^4}{(2^5)^2}$
12. $p^{-15} \times p^{15}$

13. $\frac{p^{-8} \times p^{-9}}{p^{-13} \times p^{23}}$

14. $a^{-7} \times b^{-2} \times a^{2} \times b^{-7}$

15. $\frac{a^{-21} \times a^{-3} \times a^{2}}{a^{-22} \times a^{4} \times a^{-1}}$

16. $\frac{p^{30} \times p^{35} \times p^{-50}}{p^{-17} \times p^{23}}$

17. $\frac{a^{-10}}{a^{-2}}$

18. $\frac{y^{-6}}{y^{-6}}$

19. $\frac{y^{4} \times y^{7}}{y^{15}}$

20. $\frac{a^{3} b^{-7}}{a^{-9} \times b}$

21. $\frac{m^{-5} \times m^{10}}{m^{15} \times m^{-20}}$

22. $\frac{y^{-600}}{y^{-400}} \times \frac{x^{-500}}{x^{-300}}$

23. $\frac{p^{-17} \times p^{-9}}{p^{-8} \times p^{-9}}$

24. $\frac{x^{2}}{x^{-8}}$

25. $\frac{x^{-2}}{x^{7}}$

26. $\frac{r^{10} \times p^{6}}{r^{15} \times p^{17}}$

27. $a^{-12} \times b^{17}$

28. $\frac{a^{-3} x^{-8}}{a^{-9}}$

29. $3\frac{\pi r^{2} h^{4}}{\pi r^{3} rh}$

30. $\frac{h^{-8} f^{12}}{f^{20}}$

Exponential equations

Mathematical equations which involve exponents are called exponential equations. An exponential equation is of the form $x^p = y^q$ where x, y are bases and p, q are exponents. The laws of exponents are usually used when solving equations that involve exponents. The following should be considered when solving such equations:

(a) If the bases are the same, then the exponents must be equal for the two expressions to be equal. That is, if $m^x = m^y$, then $x = y$.

(b) If the exponents are equal then the bases must be equal for the two expressions to be equal. That is, if $a^n = b^n$, then $a = b$.

Example 1.14

Find the value of n in each of the following equations:

(a) $2^{n+1} = 64$ (b) $4^n = 16$

Solution

(a) $2^{n+1} = 64$

$2^{n+1} = 2^6$ (64 expressed as an exponential number in base 2)

$n + 1 = 6$ (since the bases are the same, the exponents are equal)

$n = 6 - 1$

$n = 5$

Therefore, $n = 5$.

(b) $4^n = 16$

$4^n = 4^2$ (16 expressed as an exponential number in base 4)

n = 2 (since the bases are the same, exponents are equal)

Therefore, $n = 2$.

Example 1.15

Find the value of b in each of the following equations:

(a) $b^3 = 27$ (b) $(b + 1)^3 = 64$

Solution

(a) $b^3 = 27$

$b^3 = 3^3$ (27 expressed as an exponential number in base 3)

$b = 3$ (bases are the same therefore, the exponents are equal)

(b) $(b + 1)^3 = 64$

$(b + 1)^3 = 4^3$ (64 expressed as an exponential number in base 4)

$b + 1 = 4$ (exponents are the same so the bases are equal)

$b = 4 - 1$

$b = 3$

Therefore, $b = 3$

Exercise 1.5

Solve the following exponential equations:

1. $3a^3 = 24$
2. $2x^3 = 16$
3. $x^3 = 16$
4. $r^{-2} = 4$
5. $2^{x+1} = 2^5$
6. $x^2 = 64$
7. $\left(\frac{1}{3}\right)^x = 81^{-1}$
8. $h^2 = 0.01$
9. $(y+3)^2 = 5^2$
10. $(x^4)^2 = x^{12x}$
11. $2^x = 4^{x-3}$
12. $\left(\frac{1}{2}\right)^{-3} = 8^x$
13. $(5y)^2 = 5^2$
14. $(1-x)^{2x} = (1-x)^{\frac{1}{2}}$
15. $288 = 2x^2$

Fractional exponents

Some exponential numbers can be written with fractional exponents. By using the laws of exponents,

$$5^0 = 1 \text{ and } 5^{-1} = \frac{1}{5}$$

What does $5^{\frac{1}{2}}$ mean? By the laws of exponents it follows that:

$$\left(5^{\frac{1}{2}}\right)^2 = 5^{\frac{1}{2}\times 2} = 5^1 = 5$$

This means that, when $5^{\frac{1}{2}}$ is squared, the result is 5. Therefore, $5^{\frac{1}{2}}$ is the square root of 5.

$$\left(5^{\frac{1}{2}}\right)^2 = 5$$

Find the square root both sides of the equation to get,

$$\sqrt{\left(5^{\frac{1}{2}}\right)^2} = 5^{\frac{1}{2}} = \sqrt{5}$$

Hence, $5^{\frac{1}{2}} = \sqrt{5}$

Similarly, $\left(5^{\frac{1}{3}}\right)^3 = 5^{\frac{1}{3}\times 3} = 5^1 = 5$

Thus, $\left(5^{\frac{1}{3}}\right)^3 = 5$

Find the cube root on both sides of the equation.

Hence, $5^{\frac{1}{3}} = \sqrt[3]{5}$

Generally, if x is a positive number and n is a positive integer, then;

$$\left(x^{\frac{1}{n}}\right)^n = x^{\frac{1}{n}\times n} = x^1 = x$$

Generally, $x^{\frac{1}{n}} = \sqrt[n]{x}$, if $x > 0$.

Example 1.16

Find the value of $49^{\frac{1}{2}}$

Solution

Express 49 as a product of prime factors

$49 = 7\times 7 = 7^2$

$49^{\frac{1}{2}} = (7^2)^{\frac{1}{2}} = 7^{2\times\frac{1}{2}} = 7^1$

Therefore, $49^{\frac{1}{2}} = 7$.

Example 1.17

Simplify $\left(\frac{8}{27}\right)^{\frac{1}{3}}$

Solution

Express the numerator and denominator of $\frac{8}{27}$ as a product of prime factors.

$\frac{8}{27} = \frac{2\times 2\times 2}{3\times 3\times 3} = \frac{2^3}{3^3}$

$\left(\frac{8}{27}\right)^{\frac{1}{3}} = \left(\left(\frac{2}{3}\right)^3\right)^{\frac{1}{3}} = \frac{2}{3}$

Therefore, $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{2}{3}$.

Example 1.18

Find $(-125)^{\frac{1}{3}}$

Solution

Express -125 as a product of prime factors.

$-125 = (-5)\times(-5)\times(-5)$

$(-125)^{\frac{1}{3}} = \left((-5)^3\right)^{\frac{1}{3}} = (-5)^{3\times\frac{1}{3}} = -5$

Therefore, $(-125)^{\frac{1}{3}} = -5$.

Exercise 1.6

Simplify the following exponential numbers:

1. $(64)^{\frac{1}{2}}$
2. $(27)^{\frac{1}{3}}$
3. $(100)^{\frac{1}{2}}$
4. $(1000)^{\frac{1}{3}}$
5. $\left(\frac{1}{4}\right)^{\frac{1}{2}}$
6. $\left(\frac{81}{625}\right)^{\frac{1}{2}}$
7. $(0.01)^{\frac{1}{2}}$
8. $(0.25)^{\frac{1}{2}}$
9. $(0.027)^{\frac{1}{3}}$
10. $(16)^{\frac{1}{4}}$
11. $(81)^{\frac{1}{4}}$
12. $(32)^{\frac{1}{5}}$
13. $(1.21)^{\frac{1}{2}}$
14. $\left(\frac{27}{64}\right)^{\frac{1}{3}}$
15. $(0.001)^{\frac{1}{3}}$
16. $\left(\frac{1}{25}\right)^{\frac{1}{2}}$
17. $\left(\frac{9}{100}\right)^{\frac{1}{2}}$
18. $\left(\frac{8}{125}\right)^{\frac{1}{3}}$

Radicals

Usually a base of a given exponential number has a relationship with its power or the number which is expressed by the exponential expression.

Activity 1.2: Deducing the relationship between a base and its exponential number.

1. Copy the following chart.

Number	Expression as a product of like factors	Number of times the factor repeats	Power form of the number	Base of the power form
8	$2 \times 2 \times 2$	3	2^3	2
32				
125				
64				
$\frac{9}{49}$				
$\frac{1}{1\,000}$				

2. Fill in the blank spaces of the chart.
3. Deduce the relationship between the numbers and the bases of the power forms you have obtained.

From this activity, a number repeatedly multiplied by itself can be expressed in exponent form. In this case the exponent is a whole number.

Expressing a number in radical form

When an exponent is a fraction, the number can be expressed in radical form. Sometimes it is known as surd form. The symbol for radicals is $\sqrt[n]{\ }$, where n is called the index.

When $n = 2$, the radical is called the square root and it is written as $\sqrt{\ }$ without the index 2. Thus, the square root of 2 is written as $\sqrt{2}$.

When $n = 3$, the radical is called the cube root, and it is written as $\sqrt[3]{\ }$. Thus, the cube root of 8 is written as $\sqrt[3]{8}$.

When $n = 4$, the radical is called the fourth root, and it is written as $\sqrt[4]{\ }$. Thus, the fourth root of 81 is written as $\sqrt[4]{81}$.

When the exponent is $\frac{1}{n}$, the radical is called the n^{th} root and it is written as $\sqrt[n]{\ }$. Thus, the n^{th} root of m is written as $\sqrt[n]{m}$.

To find $\sqrt[3]{8}$, express 8 as a product of three factors, that is, $8 = 2 \times 2 \times 2$

Therefore, $\sqrt[3]{8} = \sqrt[3]{2 \times 2 \times 2}$

$$= \sqrt[3]{2^3}$$

$$= (2^3)^{\frac{1}{3}}$$

$$= 2$$

A root can also be expressed as a power. For example, $\sqrt{8}$ can be expressed in power form as follows:

Suppose $\sqrt{8} = 8^x$

$$(\sqrt{8})^2 = (8^x)^2$$

$$8 = 8^{2x}$$

$$2^3 = (2^3)2^x = 2^{6x}$$

That is, $2^3 = 2^{6x}$

Therefore, $6x = 3$

$$x = \frac{3}{6} = \frac{1}{2}$$

Therefore, $\sqrt{8}$ in power form is $8^{\frac{1}{2}}$.

Or $\quad 8^1 = 8^{2x}$

$1 = 2x$

Therefore, $x = \frac{1}{2}$

Similarly, $\sqrt[4]{81} = (81)^{\frac{1}{4}}$ and $\sqrt[5]{32} = (32)^{\frac{1}{5}}$

> In general, $\sqrt[n]{q} = (q)^{\frac{1}{n}}$. This shows that, radicals are fractional exponents.

Finding square roots and cube roots of numbers by prime factorization

It is easy to find the square root of a simple number like 4. That is, the square of 2 is 4, therefore the square root of 4 is 2. This is written as $\sqrt{4} = 2$. To find the square root of a large number, express it in terms of its prime factors. Then, write the factors in terms of exponents. If each exponent is even, then take half of each exponent to get the square root.

Example 1.19

Find the square root of 196.

Solution

Factorize 196 in terms of its prime factors as follows:

$196 = 2 \times 2 \times 7 \times 7$

$\quad = 2^2 \times 7^2$

So, $196 = 2^2 \times 7^2$

Since both exponents are even, then divide each exponent by 2.

Express the radicals in fractions and simplify as follows:

$$\sqrt{196} = \sqrt{2^2 \times 7^2}$$
$$= (2^2)^{\frac{1}{2}} \times (7^2)^{\frac{1}{2}}$$
$$= 2 \times 7$$
$$= 14$$

Therefore, $\sqrt{196} = 14$.

Note that, the square root of negative real numbers does not exist in the set of real numbers. That is, $\sqrt{-x}$ is not defined in the set of real numbers.

Example 1.20

Find the cube root of 216.

Solution

Factorise 216 in terms of its prime factors as follows:

$216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 3^3$

$$\sqrt[3]{216} = \sqrt[3]{2^3 \times 3^3}$$
$$= (2^3)^{\frac{1}{3}} \times (3^3)^{\frac{1}{3}}$$
$$= 2 \times 3$$
$$= 6$$

Therefore, the cube root of 216 is 6.

Example 1.21

Express $\sqrt[2]{1024}$ in its simplest form.

Solution

Factorise 1024 in terms of its prime factors

$1024 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

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$= \sqrt[3]{(2\times2\times2)\times(2\times2\times2)\times(2\times2\times2)\times2}$

$= \sqrt[3]{2^3\times2^3\times2^3\times2}$

$= (2^3)^{\frac{1}{3}}\times(2^3)^{\frac{1}{3}}\times(2^3)^{\frac{1}{3}}\times(2)^{\frac{1}{3}}$

$= 2\times2\times2\times2^{\frac{1}{3}}$

$= 8\sqrt[3]{2}$

Therefore, $\sqrt[3]{1024} = 8\sqrt[3]{2}$.

Example 1.22

Express each of the following numbers in its simplest form.

(a) $\sqrt{20}$

(b) $\sqrt[3]{54}$

Solution

(a) $\sqrt{20} = \sqrt{2\times2\times5}$

$= \sqrt{2\times2}\times\sqrt{5}$

$= 2\times\sqrt{5}$

$= 2\sqrt{5}$

(b) $\sqrt[3]{54} = \sqrt[3]{3\times3\times3\times2}$

$= \sqrt[3]{3\times3\times3}\times\sqrt[3]{2}$

$= 3\times\sqrt[3]{2}$

$= 3\sqrt[3]{2}$

Example 1.23

Express the following numbers under a single radical:

(a) $2\sqrt{3}$

(b) $2\sqrt[3]{4}$

Solution

(a) $2\sqrt{3} = \sqrt{2\times2\times3}$

$= \sqrt{12}$

Therefore, $2\sqrt{3} = \sqrt{12}$.

(b) $2\sqrt[3]{4} = \sqrt[3]{2\times2\times2\times4}$

$= \sqrt[3]{32}$

Therefore, $2\sqrt[3]{4} = \sqrt[3]{32}$.

Exercise 1.7

Answer the following questions:

1. Simplify each of the following radicals:

(a) $\sqrt{169}$	(b) $\sqrt{729}$	(c) $\sqrt{2048}$
(d) $\sqrt{2500}$	(e) $\sqrt{1024}$	(f) $\sqrt[3]{512}$
(g) $\sqrt[3]{343}$	(h) $\sqrt[3]{1000}$	(i) $\sqrt[3]{729}$

2. Express each of the following radicals in its simplest form.

(a) $\sqrt{40}$	(b) $\sqrt{250}$	(c) $\sqrt{1024}$
(d) $\sqrt[3]{54}$	(e) $\sqrt[3]{270}$	(f) $\sqrt[3]{162}$
(g) $\sqrt[3]{2000}$	(h) $\sqrt{2000}$	(i) $\sqrt[3]{1000000}$

3. Express each of the following numbers under a single radical sign:

(a) $5\sqrt{2}$	(b) $4\sqrt{11}$	(c) $3\sqrt{10}$
(d) $9\sqrt{3}$	(e) $5\sqrt[3]{2}$	(f) $3\sqrt[3]{3}$
(g) $2\sqrt[3]{(-1000)}$	(h) $6\sqrt[3]{4}$	(i) $7\sqrt[3]{5}$

Addition and subtraction of radicals

Two or more radicals can be added or subtracted if they are alike. Radicals which are alike are those with the same indices. They are added or subtracted in the same way as normal numbers. This means that only radicals of the same index can be added or subtracted, just as is done with algebraic expressions. Therefore, the radicals $\sqrt{2}$ and $\sqrt[3]{2}$ cannot be added or subtracted. Before adding or subtracting radicals, first simplify the terms if possible.

Activity 1.3: Deducing the conditions for adding and subtracting radicals.

1. In pairs, simplify the following radicals: $\sqrt{8}$, $\sqrt{32}$, $\sqrt{27}$ and $\sqrt{3\times2\times2}$.
2. Simplify the given radicals, and identify the like terms.
3. Consider the like terms and express each term as a sum of its roots.
4. For each group of like terms add the values obtained in step 3.
5. Pick any two terms of unlike radicals and try to add them as in step 4. What can you conclude?
6. Provide the condition for radicals to be added together.

Example 1.24

Simplify each of the following radicals:

(a) $\sqrt{2}+\sqrt{2}$ (b) $2\sqrt{3}+3\sqrt{3}$ (c) $\sqrt{8}+\sqrt{32}$

Solution

(a) $\sqrt{2}+\sqrt{2}=1\sqrt{2}+1\sqrt{2}=2\sqrt{2}$ (b) $2\sqrt{3}+3\sqrt{3}=5\sqrt{3}$

(c) $\sqrt{8}=\sqrt{2\times2\times2}=2\sqrt{2}$

$\sqrt{32}=\sqrt{2\times2\times2\times2\times2}=4\sqrt{2}$

Thus, $\sqrt{8}+\sqrt{32}=\sqrt{2\times2\times2}\times\sqrt{2\times2\times2\times2\times2}$

$=2\sqrt{2}+4\sqrt{2}$

$=6\sqrt{2}$

Example 1.25

Simplify each of the following radicals:

(a) $6\sqrt{7}-2\sqrt{7}$ (b) $6\sqrt{28}-2\sqrt{63}$ (c) $\sqrt{32}-\sqrt{8}$

Solution

(a) $6\sqrt{7}-2\sqrt{7}=4\sqrt{7}$

(b) $6\sqrt{28}-2\sqrt{63}$

Simplify $\sqrt{28}$ and $\sqrt{63}$ as follows:

$\sqrt{28}=\sqrt{4\times 7}=2\sqrt{7}$ and $\sqrt{63}=\sqrt{9\times 7}=3\sqrt{7}$

$6\sqrt{28}-2\sqrt{63}=6\times 2\sqrt{7}-2\times 3\sqrt{7}$

$6\sqrt{28}-2\sqrt{63}=12\sqrt{7}-6\sqrt{7}$

$=6\sqrt{7}$

(c) $\sqrt{32}-\sqrt{8}$

$=4\sqrt{2}-2\sqrt{2}$

$=2\sqrt{2}$

Exercise 1.8

Simplify the following radicals:

1. $\sqrt{27}+5\sqrt{3}$
2. $\sqrt{44}+\sqrt{11}$
3. $\sqrt{120}+\sqrt{1080}$
4. $125+3\sqrt{120}$
5. $\sqrt{125}-\sqrt{45}$
6. $\sqrt{45}+\sqrt{125}$
7. $\sqrt{5}+2\sqrt{20}$
8. $\sqrt{15}+\sqrt{90}+\sqrt{135}$
9. $\sqrt{162}-\sqrt{216}$
10. $2\sqrt{50}-\sqrt{8}$
11. $5\sqrt{8}-6\sqrt{2}$
12. $\sqrt{243}-\sqrt{27}$
13. $\sqrt{162}-\sqrt{108}$
14. $5\sqrt{28}-6\sqrt{63}+\sqrt{112}$
15. $\sqrt{128}-\sqrt{32}$
16. $2\sqrt{3}+\frac{2}{3}\sqrt{27}$
17. $\frac{1}{2}\sqrt{112}-\frac{5}{6}\sqrt{63}$
18. $6\sqrt{32}-\sqrt{8}$
19. $8\sqrt{150}-2\sqrt{96}-3\sqrt{24}$
20. $3\sqrt{60}-\sqrt{15}+2\sqrt{60}$

Multiplication of radicals

When multiplying two or more radicals of the same index, express each radical in its simplest form. Then, multiply numbers which are inside by numbers outside the radical sign and then simplify where possible.

Example 1.26

Simplify each of the following radical products:

(a) $\sqrt{3}\times\sqrt{5}$ (b) $\sqrt{2}\times\sqrt{32}$ (c) $\sqrt{20}\times\sqrt{28}$

(d) $\sqrt{12}\left(\sqrt{3}+\sqrt{5}\right)$ (e) $\left(2\sqrt{3}-\sqrt{2}\right)\times\left(\sqrt{3}+3\sqrt{2}\right)$

Solution

(a) $\sqrt{3}\times\sqrt{5}=\sqrt{3\times5}$

(both the multiplicand and multiplier are of the same index)

$=\sqrt{15}$

Therefore, $\sqrt{3}\times\sqrt{5}=\sqrt{15}$

(b) Simplify $\sqrt{32}\times\sqrt{2}$

$$\sqrt{32}=\sqrt{2\times2\times2\times2\times2}$$
$$=4\sqrt{2}$$
$$\sqrt{2}\times\sqrt{32}=\sqrt{2}\times4\sqrt{2}$$
$$=1\times4\times\sqrt{2}\times\sqrt{2}$$
$$=1\times4\times2$$
$$=8$$

Therefore, $\sqrt{2}\times\sqrt{32}=8$

(c) Simplify $\sqrt{20}$ and $\sqrt{28}$ as follows:

$$\sqrt{20} = \sqrt{2\times2\times5} = 2\sqrt{5}$$
$$\sqrt{28} = \sqrt{2\times2\times7} = 2\sqrt{7}$$

Thus, $\sqrt{20}\times\sqrt{28} = 2\sqrt{5}\times2\sqrt{7}$

$$= 2\times2\times\sqrt{5}\times\sqrt{7}$$
$$= 4\times\sqrt{5\times7}$$
$$= 4\sqrt{35}$$

Therefore, $\sqrt{20}\times\sqrt{28} = 4\sqrt{35}$.

(d) $\sqrt{12}\left(\sqrt{3}+\sqrt{5}\right) = \sqrt{4\times3}\left(\sqrt{3}+\sqrt{5}\right)$ by expanding $\sqrt{12}$

$= 2\sqrt{3}\left(\sqrt{3}+\sqrt{5}\right)$ by simplifying $\sqrt{4\times3}$

$= 2\sqrt{3}\times\sqrt{3} + 2\sqrt{3}\times\sqrt{5}$ by opening brackets

$= (2\times3) + 2\sqrt{15}$

$= 6 + 2\sqrt{15}$

Therefore, $\sqrt{12}\left(\sqrt{3}+\sqrt{5}\right) = 6 + 2\sqrt{15}$.

(e) By opening the brackets,

$$\left(2\sqrt{3}-\sqrt{2}\right)\times\left(\sqrt{3}+3\sqrt{2}\right) = \left(2\sqrt{3}\times\sqrt{3}\right)+\left(2\sqrt{3}\times3\sqrt{2}\right)+\left(-\sqrt{2}\times\sqrt{3}\right)+\left(-\sqrt{2}\times3\sqrt{2}\right)$$
$$= 2\sqrt{9}+6\sqrt{6}-\sqrt{6}-3\sqrt{4}$$
$$= 2\times3+5\sqrt{6}-3\times2$$
$$= 6+5\sqrt{6}-6$$
$$= 6-6+5\sqrt{6}$$
$$= 5\sqrt{6}$$

Therefore, $\left(2\sqrt{3}-\sqrt{2}\right)\times\left(\sqrt{3}+3\sqrt{2}\right) = 5\sqrt{6}$.

In general, $\sqrt[n]{a}\times\sqrt[n]{b} = \sqrt[n]{ab}$

Exercise 1.9

Simplify each of the radicals in the following expressions and perform the indicated operations:

1. $\sqrt{20}\times\sqrt{5}$
2. $\sqrt{12}\times\sqrt{3}$
3. $\sqrt{45}\times\sqrt{54}$
4. $2\sqrt{3}\times\sqrt{54}$
5. $3\sqrt{10}\times3\sqrt{10}$
6. $\sqrt{32}\times\sqrt{12}$
7. $2\sqrt{3}\times5\sqrt{12}$
8. $\left(2\sqrt{15}\right)^2$
9. $\left(\sqrt{6}\right)^3$
10. $\left(2\sqrt{15}\right)^2$
11. $\left(\sqrt{5}\right)^3\times\left(\sqrt{13}\right)^2$
12. $\left(\sqrt{2}+1\right)^2$
13. $2\sqrt{2}\times\sqrt{10}\times\sqrt{20}$
14. $\sqrt{6}\times\sqrt{8}\times\sqrt{10}\times\sqrt{12}$
15. $\sqrt{5}\times\sqrt{24}\times\sqrt{40}$
16. $\sqrt{8}\left(\sqrt{2}-\sqrt{18}\right)$
17. $\sqrt{5}\left(\sqrt{2}+\sqrt{18}\right)$
18. $6\sqrt{2}\left(\sqrt{2}-\sqrt{18}\right)$
19. $\left(2\sqrt{5}+1\right)\times\left(3\sqrt{5}+1\right)$
20. $\left(\sqrt{8}-\sqrt{2}\right)^2$
21. $\left(\sqrt{7}+\sqrt{5}\right)\left(\sqrt{7}-\sqrt{5}\right)$
22. $\left(4\sqrt{3}-\sqrt{12}\right)\left(\sqrt{3}+\sqrt{2}\right)$
23. $\left(2\sqrt{3}+1\right)^2$
24. $\left(2\sqrt{3}-\sqrt{5}\right)\left(2\sqrt{3}+3\sqrt{5}\right)$

Division of radicals

Two radicals can be divided by writing the divisor under the dividend in the form of a fraction. If two numbers are separately under the same radical, then divide the numbers under one radical and simplify. But if the divisor and dividend have different radicals, it is not possible for the numbers to be under one radical.

Activity 1.4: Deducing conditions for dividing radicals

1. In pairs, follow the steps given to divide $\sqrt{20}$ by $\sqrt{5}$.
2. Express $\sqrt{20} \div \sqrt{5}$ as a ratio of numerator and denominator.
3. Express the numerator and denominator in simple terms.
4. Collect together the terms with the same radicals and divide.
5. Follow the same steps to evaluate $\sqrt{625} \div 2\sqrt{25}$.

Example 1.27

Express the following radicals in their most simplified form:

(a) $\sqrt{75} \div \sqrt{12}$ (b) $5\sqrt{\frac{18}{50}}$ (c) $\frac{6\sqrt{5} \times 2\sqrt{3}}{\sqrt{20} \times 3\sqrt{21}}$

Solution

(a) $\sqrt{75} \div \sqrt{12} = \frac{\sqrt{75}}{\sqrt{12}}$ (divisor and dividend in the fraction form)

$= \sqrt{\frac{75}{12}}$ (divisor and dividend have the same root)

$= \sqrt{\frac{25}{4}}$ (simplified fraction)

$= \frac{\sqrt{25}}{\sqrt{4}}$

$= \frac{5}{2}$.

(b) $5\sqrt{\frac{18}{50}} = 5 \times \sqrt{\frac{18}{50}}$

$= 5 \times \frac{\sqrt{18}}{\sqrt{50}}$

$= 5 \times \frac{3\sqrt{2}}{5\sqrt{2}}$

$= 5 \times \frac{3 \times \sqrt{2}}{5 \times \sqrt{2}}$

$= 3.$

(c) $\frac{6\sqrt{5} \times 2\sqrt{3}}{\sqrt{20} \times 3\sqrt{12}} = \frac{6\sqrt{5} \times 2\sqrt{3}}{\sqrt{(4 \times 5)} \times 3\left(\sqrt{4 \times 3}\right)}$

$= \frac{6\sqrt{5} \times 2\sqrt{3}}{2\sqrt{5} \times 3 \times 2\sqrt{3}}$

$= \frac{3}{3}$

$= 1.$

Generally $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ where a and b are positive numbers.

Exercise 1.10

Simplify each of the following radical expressions:

1. $\frac{\sqrt{50}}{\sqrt{2}}$
2. $\sqrt{\frac{75}{3}}$
3. $\frac{\sqrt{12}}{\sqrt{50}}$
4. $\frac{\sqrt{121}}{\sqrt{44}}$
5. $\sqrt{\frac{225}{3}}$
6. $\frac{\sqrt{72}}{\sqrt{900}}$
7. $\frac{2\sqrt{7}}{\sqrt{21}}$
8. $\frac{3\sqrt{5}}{\sqrt{45}}$
9. $\frac{\sqrt{18}}{2\sqrt{6}}$
10. $\frac{3\sqrt{50}}{5\sqrt{32}}$
11. $\frac{\sqrt{8} \times \sqrt{2}}{\sqrt{12} \times \sqrt{3}}$
12. $\frac{\sqrt{14} \times 2\sqrt{3}}{\sqrt{12} \times \sqrt{56}}$
13. $\frac{5\sqrt{7} \times 2\sqrt{3}}{\sqrt{45} \times \sqrt{21}}$
14. $\frac{\sqrt{14} \times 2\sqrt{3}}{\sqrt{48} \times \sqrt{28}}$
15. $\frac{\sqrt{18} \times \sqrt{20} \times \sqrt{24}}{\sqrt{8} \times \sqrt{30}}$

Rationalising the denominator

Rationalising the denominator is the process of eliminating any radical expression in the denominator. Rationalising the denominator involves multiplication of the denominator by a suitable radical resulting in a denominator which has no radical. The best choice for the radical is obtained by considering the following:

(i) If the denominator is a single radical term, the most suitable choice is a radical itself.

For example, rationalising the denominator of $\frac{a}{\sqrt{b}}$.

Choose $\sqrt{b}$ to be a rationalizing factor.

Multiply both the numerator and denominator by $\sqrt{b}$ as follows:

$$\frac{a}{\sqrt{b}}=\frac{a}{\sqrt{b}}\times\frac{\sqrt{b}}{\sqrt{b}}=\frac{a\sqrt{b}}{b}$$

Therefore, $\frac{a}{\sqrt{b}}=\frac{a\sqrt{b}}{b}$.

(ii) If a denominator has a radical term involving the (+) or (–) operations signs, the most suitable choice of a rationalizing factor is the same denominator expression, but with the operation sign changed either from (+) to (–) or changed from (–) to (+).

For example, rationalising the denominator of $\frac{1}{\sqrt{a}+\sqrt{b}}$.

Choose $\sqrt{a}-\sqrt{b}$ to be a rationalising factor.

Multiply both numerator and denominator by $\sqrt{a}-\sqrt{b}$ as follows:

$$\frac{1}{\sqrt{a}+\sqrt{b}}=\frac{1}{\sqrt{a}+\sqrt{b}}\times\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{a}-\sqrt{b}}{a-b}$$

$$\frac{1}{\sqrt{a}-\sqrt{b}}=\frac{1}{\sqrt{a}-\sqrt{b}}\times\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{\sqrt{a}+\sqrt{b}}{a-b}$$

Therefore, $\frac{1}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{a}+\sqrt{b}}{a-b}$

Based on these two examples, the best choice of radical expression in the following examples is as follows:

Denominator	Rationalizing factor	Reason
$\sqrt{a}$	$\sqrt{a}$	$\sqrt{a}\times\sqrt{a}=a$
$\sqrt{a}-\sqrt{a}$	$\sqrt{a}+\sqrt{a}$	$\left(\sqrt{a}-\sqrt{a}\right)\left(\sqrt{a}+\sqrt{a}\right)=a-b$
$\sqrt{a}+\sqrt{a}$	$\sqrt{a}-\sqrt{a}$	$\left(\sqrt{a}+\sqrt{a}\right)\left(\sqrt{a}-\sqrt{a}\right)=a-b$

Example 1.28

Rationalize the denominator in each of the following expressions:

(a) $\frac{3}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{5}-\sqrt{3}}$

(c) $\frac{\sqrt{5}}{\sqrt{5}+\sqrt{3}}$ (d) $\frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{5}}$

Solution

(a) $\sqrt{5}$ is a denominator and a single radical expression.

Multiply both numerator and denominator by $\sqrt{5}$

$$\frac{3}{\sqrt{5}}=\frac{3}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}$$

$$=\frac{3\sqrt{5}}{5}$$

$$\therefore\ \frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{5}.$$

(b) $\sqrt{5}-\sqrt{3}$ is a denominator involving subtraction.

Change the subtraction sign in $\sqrt{5}-\sqrt{3}$ to addition sign to obtain $\sqrt{5}+\sqrt{3}$ as a rationalising factor.

Multiply both numerator and denominator by $\sqrt{5}+\sqrt{3}$ as follows:

$$\frac{1}{\sqrt{5}-\sqrt{3}}=\frac{1}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{\sqrt{5}+\sqrt{3}}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}$$

$$=\frac{\sqrt{5}+\sqrt{3}}{5-3}$$

$$=\frac{\sqrt{5}+\sqrt{3}}{2}.$$

(c) $\sqrt{5}+\sqrt{3}$ is the denominator involving addition sign.

Change the addition sign to subtraction to obtain $\sqrt{5}-\sqrt{3}$ as rationalizing factor. Multiply both numerator and denominator by $\sqrt{5}-\sqrt{3}$ as follows:

$$\frac{\sqrt{5}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{5}}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$$

$$=\frac{\sqrt{5}\left(\sqrt{5}-\sqrt{3}\right)}{5-3}$$

$$=\frac{5-\sqrt{15}}{2}$$

(d) If $\sqrt{2}-\sqrt{5}$ is the denominator, $\sqrt{2}+\sqrt{5}$ is the rationalizing factor. Multiply both numerator and denominator by $\sqrt{2}+\sqrt{5}$ as follows:

$$\frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{5}} = \frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{5}} \times \frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}}$$

$$= \frac{\left(2+\sqrt{3}\right)\left(\sqrt{2}+\sqrt{5}\right)}{\left(\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+\sqrt{5}\right)}$$

$$= \frac{2\sqrt{2}+2\sqrt{5}+\sqrt{6}+\sqrt{15}}{2-5}$$

$$= \frac{2\sqrt{2}+2\sqrt{5}+\sqrt{6}+\sqrt{15}}{-3}$$

Exercise 1.11

Simplify each of the following expressions by rationalising the denominator:

1. $\frac{1}{\sqrt{5}}$
2. $\frac{1}{3\sqrt{8}}$
3. $\frac{\sqrt{3}}{\sqrt{18}}$
4. $\frac{2\sqrt{30}}{\sqrt{7}}$
5. $\frac{1}{5+\sqrt{10}}$
6. $\frac{1+\sqrt{2}}{\sqrt{2}}$
7. $\frac{\sqrt{7}+8}{\sqrt{3}}$
8. $\frac{\sqrt{6}+3\sqrt{2}}{\sqrt{2}}$
9. $\frac{\sqrt{8}}{\sqrt{2}-\sqrt{3}}$
10. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$
11. $\frac{5-\sqrt{6}}{\sqrt{7}-\sqrt{6}}$
12. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

13. $\frac{9\sqrt{3}-\sqrt{2}}{\sqrt{5}+3\sqrt{2}}$

14. $\frac{c}{5\sqrt{a}+\sqrt{b}}$

15. $\frac{x-2y}{\sqrt{x}+\sqrt{y}}$

16. $\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}$

17. $\frac{2}{\sqrt{5}-3}+\frac{1}{1-2\sqrt{3}}$

18. $\frac{\sqrt{8}}{4-\sqrt{5}}+\frac{1}{7-2\sqrt{3}}$

19. $\frac{2\sqrt{7}}{\sqrt{10}-\sqrt{7}}-\frac{1}{\sqrt{3}+\sqrt{7}}$

20. $\left(2+\sqrt{2}\right)\div\left(\sqrt{3}-\sqrt{2}\right)$

Finding square roots and cube roots of numbers using mathematical tables

The square roots and cube roots of numbers can be obtained from mathematical tables.

Square roots

Before taking a reading from tables of square roots, first estimate the square root of the given number by grouping its digits in pairs from the right hand side.

For example, to find the square root of 196 from Table 1.1, first group the digits in pairs from the right, that is, group the digits of 196 as 1' 96. Then, estimate the square root of a digit or pair of digits on the extreme left. In this case it is 1.

The square root of 1 is 1. Since there are two groups of digits, the square root of 196 will be a number of two digits before the decimal point referred as mean difference in the tables. Given the number 196, locate 1.9 in the table of square roots on the extreme left. Look at the column labeled 6. It meets the row 1.9 at 1.400. Therefore the square root of 196 is 14.

In the same way the square root of 7.578 can be found from Table 1.1 by first locating 7.5 in column x, then finding the number where the column with 7 meets the row with 7.5 which is 2.751. Add the number obtained under column 8 of the mean difference to the decimal part of 2.751 that is 751 + 1 = 752. Therefore, the square root of 7.578 is 2.752.

Table 1.1: Square roots of numbers from 1 to 10

$\sqrt{x}$								Mean Differences (Add)								
x	0	1	...	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.000	1.005		1.030	1.034	1.039	1.044	0	1	1	2	2	3	3	4	4
1.1	1.049	1.054		1.077	1.082	1.086	1.091	0	1	1	2	2	3	3	4	4
⋮																
1.8	1.342	1.345		1.364	1.367	1.371	1.375	0	1	1	1	2	2	3	3	3
1.9	1.378	1.382		1.400	1.404	1.407	1.411	0	1	1	1	2	2	3	3	3
⋮																
7.4	2.720	2.722	2.724	2.731	2.733	2.735	2.737	0	0	1	1	1	1	1	1	2
7.5	2.739	2.740	2.742	2.750	2.751	2.753	2.755	0	0	1	1	1	1	1	1	2
7.6	2.757	2.759	2.760	2.768	2.769	2.771	2.773	0	0	1	1	1	1	1	1	2

Similarly, to find the square root of 5678 from Table 1.2, first group the digits in pairs from the right, that is 56'78. The square root of 56 is between 7 and 8. Therefore, the square root of 5678 starts with 7 and has two digits before the decimal point. As shown in Table 1.2, the number 7.530 is obtained at a place where the row with 56 on the extreme left meets the column labeled 7.

Similarly, 5 is read where the column labeled 8 (within the mean difference) meets the row with 56 on the extreme left. Add the 5 to the last digit of 7.530 to obtain 7.535. Therefore, the square root of 5678 is 75.35.

Table 1.2: Square roots

$\sqrt{x}$									Mean Differences (Add)								
x	0	1	2	...	6	7	8	9	1	2	3	4	5	6	7	8	9
10	3.162	3.178	3.194		3.256	3.271	3.286	3.302	2	3	5	6	8	9	11	12	14
11	3.317	3.332	3.347		3.406	3.421	3.435	3.450	1	3	4	6	7	9	10	12	13
. . .																	
55	7.416	7.423	7.430		7.457	7.463	7.470	7.477	1	1	2	3	3	4	5	5	6
56	7.483	7.490	7.497		7.523	7.530	7.537	7.543	1	1	2	3	3	4	5	5	6
57	7.550	7.556	7.563		7.589	7.596	7.603	7.609	1	1	2	3	3	4	5	5	6
. . .																	

To find the square root of a number with more than four digits, first round off the number to four significant figures. For example, to find the square root of 75678 (which is a five digits number), first round it off 75678 to four significant digits, to obtain 75680. Then, pair the digits from the right and estimate the square root of the number in the group on the extreme left and determine the position of the decimal point. Therefore, 7'56'80, shows that the square root has three digits before the decimal point. The square root of 7 is between 2 and 3. By using the square root table, the answer is 275.1. (see Table 1.1).

Exercise 1.12

Use mathematical tables to find each of the following radicals:

1. $\sqrt{2156}$	2. $\sqrt{1024}$	3. $\sqrt{267}$	4. $\sqrt{6.74}$
5. $\sqrt{0.57}$	6. $\sqrt{89.105}$	7. $\sqrt{0.006}$	8. $\sqrt{0.0008}$
9. $\sqrt{25679}$	10. $\sqrt{3567}$	11. $\sqrt{646}$	12. $\sqrt{24}$
13. $\sqrt{154}$	14. $\sqrt{5}$	15. $\sqrt{1.44}$	16. $\sqrt{99}$

Cube roots

The cube root of a number say y is denoted by $\sqrt[3]{y}$. This is a value which results from its original number after being multiplied by itself, three times. The cube root of a number is basically the root of a natural number which is cubed. It is the reverse process of evaluating the cube of a number. Say, $x^3 = y$, then $x = \sqrt[3]{y}$. Since $4 \times 4 \times 4 = 64$, then 4 is the cube root of 64, that is $\sqrt[3]{64} = 4$. However, this approach is not possible for all real numbers. Alternatively, mathematical tables are used to find cube roots of numbers.

Finding cube roots of numbers using mathematical tables

Mathematical tables usually provide values of cube roots in four digits. The first column (labelled x) of a mathematical table contains the numbers whose cube roots are to be found as shown in Table 1.3.

Table 1.3: Cube roots of numbers

$\sqrt[3]{x}$									Mean Differences (Add)								
x	0	1	2	3	4	5	...	9	1	2	3	4	5	6	7	8	9
1.0	1.0000	1.0033	1.0066	1.0099	1.0132	1.0164		1.0291	3	6	10	13	16	19	23	26	29
1.1	1.0323	1.0354	1.0385	1.0416	1.0446	1.0477		1.0597	3	6	9	12	15	18	21	24	27
⋮																	
7.6	1.9661	1.9670	1.9678	1.9687	1.9695	1.9704		1.9738	1	2	3	3	4	5	6	7	8
7.7	1.9747	1.9755	1.9764	1.9772	1.9781	1.9789		1.9823	1	2	3	3	4	5	6	7	8
⋮																	
8.0	2.0000	2.0008	2.0017	2.0025	2.0033	2.0042		2.0075	1	2	2	3	4	5	6	7	7
8.1	2.0083	2.0091	2.0100	2.0108	2.0116	2.0124		2.0157	1	2	2	3	4	5	6	7	7
8.2	2.0165	2.0173	2.0182	2.0190	2.0198	2.0206		2.0239	1	2	2	3	4	5	6	7	7
8.3	2.0247	2.0255	2.0263	2.0271	2.0279	2.0288		2.0320	1	2	2	3	4	5	6	6	7

To find the cube root of 8, look in the first column to find 8.0. The cube root of 8.0 is found under column 0, which is 2.0000. Also, to find the cube root of 8.25, look in the first column to find 8.2, then read the answer under column 5 along the row of 8.2. The answer is 2.0206. Likewise, to find the cube root of 8.258, add the common difference obtained under column 8 along the row of 8.2 to the decimal part of 2.0206. That is .0206 + 7 = .0213, therefore the cube root of 8.258 is 2.0213.

Example 1.29

Find the cube root of 7.67.

Solution

Find 7.6 in the first column. Then, read the answer under column of 7 along the row of 7.6. The reading under column 7 is 1.9713.

Therefore $\sqrt[3]{7.67} = 1.9713$.

Example 1.30

Find $\sqrt[3]{50.92}$.

Solution

Use tables of cube roots from mathematical tables or tables appended in this book. Find 50 in the first column of a table. Then, read the number under column 9 along the row of 50, that is 3.7060.

Find the number obtained under column 2 of the common difference.

The common difference is 5. Add 5 to the decimal part of 3.7060.

That is, .7060 + 5 = .7065

Thus, the cube root of 50.92 is 3.7065.

Therefore, $\sqrt[3]{50.92} = 3.7065$.

Exercise 1.13

Answer the following questions:

1. Use mathematical tables to find the cube root of each of the following numbers:
   (a) 19 (b) 64 (c) 48.23 (d) 6.273
2. Use mathematical tables to evaluate the following numbers:
   (a) $\sqrt[3]{55.94}$ (b) $\sqrt[3]{89.4}$ (c) $\sqrt[3]{1.25}$ (d) $\sqrt[3]{3.7}$
3. The volume of a solid cube is 1.674 m^3. Find the length of its side.

Transposition of formulae

A formula is an equation which shows how variables are related. For example, the formula for the circumference of a circle is $C = \pi d$, where C is expressed in terms of π and d. Here C is the subject of the formula. It is also possible to express d as the subject of the formula. That is, $d = \frac{C}{\pi}$.

The procedure of expressing a formula in different ways is called **transposition of formula.** The following are some examples of formulae:

(a) $A = lb$ (b) $V = \pi r^2 h$ (c) $I = \frac{PRT}{100}$

(d) $A = \frac{1}{2}(a+b)h$ (e) $T = 2\pi\sqrt{\frac{l}{g}}$ (f) $y = mx + c$

Example 1.31

Given that $y = mx + c$, make m the subject of the formula.

Solution

Given the formula $y = mx + c$ (1)

Subtracting c from both sides of equation (1) results to $y - c = mx$ (2)

Dividing both sides of equation (2) by x results to $\frac{y-c}{x} = m$

Therefore, $m = \frac{y-c}{x}$ provided $x \neq 0$

Example 1.32

The volume (V) of a cylinder with a base of radius r and height h is given by $V = \pi r^2 h$. Make r the subject of the formula.

Solution

The formula $V = \pi r^2 h$ (1)

Dividing both sides of (1) by πh results to $\frac{V}{\pi h} = r^2$ (2)

Taking the square root of (2) on both sides $\sqrt{\frac{V}{\pi h}} = r$

Therefore, $r = \sqrt{\frac{V}{\pi h}}$.

Example 1.33

Given that $T = 2\pi\sqrt{\frac{l}{g}}$, make l the subject of the formula.

Solution

The formula $T = 2\pi\sqrt{\frac{l}{g}}$ (1)

Dividing both sides of (1) by 2π results to $\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$ (2)

by squaring both sides of (2): $\left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$ (3)

multiplying both sides of (3) by g: $g\left(\frac{T}{2\pi}\right)^2 = l$

but $g\left(\frac{T}{2\pi}\right)^2 = \frac{gT^2}{4\pi^2}$

Therefore, $l = \frac{gT^2}{4\pi^2}$.

Example 1.34

The simple interest (I) on principal (P) for time (T) years and at the rate of R% per annum is given by the formula $I = \frac{PRT}{100}$. Make P the subject of the formula.

Solution

formula $I = \frac{PRT}{100}$ (1)

multiplying both sides of (1) by 100: $100I = PRT$ (2)

dividing both sides of (2) by R: $\frac{100I}{TR} = P$ (3)

Therefore, $P = \frac{100I}{RT}$.

Exercise 1.14

In questions 1 to 13 make the given letter the subject in each formula.

	Formula	Letter		Formula	Letter
1.	$I = \frac{PRT}{100}$	R	2.	$C = 2\pi r$	r
3.	$A = \frac{1}{2}bh$	h	4.	$E = \frac{w}{pv}$	p
5.	$A = P + \frac{PRT}{100}$	P	6.	$A = 2\pi r(r+h)$	h
7.	$T = \frac{3}{2}\sqrt{l}$	l	8.	$V = \frac{1}{2}\pi r^2 h$	r
9.	$S = \frac{1}{2}at^2$	t	10.	$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$	f
11.	$P = w\left(\frac{1+a}{1-a}\right)$	a	12.	$T = 2\pi\sqrt{\frac{l}{g}}$	g
13.	$C = \frac{5}{9}(F-32)$	F			

Chapter summary

1. The laws of exponents are:

   (a) $x^a \times x^b = x^{a+b}$ (b) $\frac{x^a}{x^b} = x^{a-b}$

   (c) $\left(x^a\right)^b = x^{ab}$ (d) $x^{-a} = \frac{1}{x^a}$

   (e) $x^{\left(\frac{1}{a}\right)} = \sqrt[a]{x}$ (f) $x^{\frac{a}{b}} = \sqrt[b]{x^a}$ *or* $\left(\sqrt[b]{x}\right)^a$

2. The word radical means the n^{th} root, where $n = 2, 3, 4, \ldots$
   For example, if $n = 2$, we have the square root, $n = 3$ we have the cube root.

3. Numbers having the same radical signs can be added or subtracted.
   For example $8\sqrt{7}$ and $3\sqrt{7}$ have same radical. Therefore, they can be added or subtracted. For instance, $8\sqrt{7} + 3\sqrt{7} = 11\sqrt{7}$ or $8\sqrt{7} - 3\sqrt{7} = 5\sqrt{7}$.

4. In rationalizing the denominator, multiply both the numerator and denominator by the rationalizing factor. The radical is a better rationalizing factor.

5. A formula is an equation which shows the relationship of symbols.

6. To change the subject of a formula solve the equation for the letter that is to be the new subject of the formula.

Revision exercise 1

Answer the following questions:

1. Write each of the following numbers without radicals:

   (a) $\sqrt{900}$ (b) $\sqrt{160000}$ (b) $\sqrt[3]{8\times27\times5^3}$

2. Write each of the following numbers using a radical sign:

   (a) $7^{\frac{1}{2}}$ (b) $19^{\frac{2}{3}}$ (c) $2^{\frac{1}{3}}$

3. Simplify each of the following expressions:

   (a) $\sqrt{675}+\sqrt{75}$ (b) $\sqrt{1024}+\sqrt{4}$

   (c) $\sqrt[3]{8}+\sqrt{64}$ (d) $\sqrt{175}+\sqrt{28}-\sqrt{63}$

   (e) $\sqrt{\frac{1}{2}}+2\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{8}}$ (f) $\sqrt{1000}-\sqrt{40}-\sqrt{63}$

   (g) $\sqrt{25}\times\sqrt{6}$ (h) $\sqrt{75}\times\sqrt{3}$

4. Simply each of the following radicals by making the number in the radical sign as small as possible:

   (a) $\sqrt{50}$ (b) $\sqrt{375}$ (c) $\sqrt{125}$ (d) $\sqrt[3]{250}$

   (e) $\sqrt{4096}$ (f) $\sqrt{1024}$ (g) $\sqrt{729}$ (h) $\sqrt[3]{625}$

   (i) $\sqrt{1296}$ (j) $\sqrt{3000}$

5. Simplify each of the following radicals:

   (a) $\sqrt{4y^2}$ (b) $\sqrt{8ym^3}$ (c) $\sqrt{24y^3}$

   (d) $\sqrt{xy^2}$ (e) $\sqrt{729a^3b^3c^3}$

6. Rationalize the denominator in each of the following expressions and simplify:

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{2}+1}$ (c) $\frac{1}{(\sqrt{3}+1)}$ (d) $\frac{1}{\sqrt{3}+\sqrt{2}}$

(e) $\frac{2}{\sqrt{5}-1}$ (f) $\frac{\sqrt{3}+1}{2\sqrt{3}}$ (g) $\frac{\sqrt{6}+4}{\sqrt{6}+\sqrt{2}}$

7. Simplify: $\frac{(\sqrt{2}+\sqrt{3})}{(\sqrt{3}-\sqrt{2})} \div \frac{(\sqrt{2}+\sqrt{3})}{(\sqrt{3}-\sqrt{2})}$

8. Expand each of the following:

(a) $(\sqrt{3}-1)^2$ (b) $(\sqrt{5}-\sqrt{2})(\sqrt{3}-2)$ (c) $(\sqrt{3}+1)^2$

9. Find the square root of each of the following:

(a) 2916 (b) 5625 (c) 0.25

10. Use mathematical tables to evaluate each of the following:

(a) $\sqrt{1256}$ (b) $\sqrt{0.0015}$ (c) $\sqrt{256789}$

(d) $\sqrt{75}$ (e) $\sqrt{0.009}$

11. Given that $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$, write v as the subject of the formula.

12. If $v^2+u^2-2as=0$, write u as a subject of the formula.

13. Given the formula $s=ut+\frac{3}{4}at^2$, express u in terms of other letters.

14. If $s = u + at$, express u in terms of a, t and s.

15. A formula connecting u, v and f for a spherical mirror is $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$. Calculate the value of v when $f = 8.1$ and $u = 5.4$.

Chapter Two

Algebra

Introduction

Algebra is a branch of mathematics which deals with manipulation of symbols and numbers. In algebra, letters or symbols are used to represent numbers in forming and manipulating mathematical expressions. In this chapter, you will learn binary operations, brackets in computation, quadratic expressions, and factorisation. The competences developed in this chapter will enable you to find solutions of real – life problems formulated in mathematical language using information available on the problem. For example, you can find the speed of a car if you know the time it passed through two different points.

Binary operations

A binary operation is an operation that applies to two quantities or expressions. Binary operations make use of symbols that represent one or more operations, such as addition, subtraction, multiplication and division.

The symbols used in binary operations are not standard, this means, they do not represent a specific operation. A symbol may be used in a certain operation in an expression, and yet have a different meaning in another expression.

A binary operation may be denoted by symbols such as Δ, $*$ and so on, depending on the instructions given for the operation. The instructions may be given in words or by symbols.

When two numbers are added or multiplied together in order to obtain one number, you perform a binary operation (*bi* means two). Binary operations may involve the basic arithmetic operations such as addition (+), subtraction (–), multiplication (×), and division (÷) depending on the definition given.

Example 2.1

If $a * b = 5a - b$, find $6 * 9$.

Solution

$a * b = 5a - b$

$6 * 9 = 5 \times 6 - 9$

$= 30 - 9$

$= 21$

Therefore, $6 * 9 = 21$.

Example 2.2

If $a * b = a^2 - b$, find y given that $4 *(2 * y) = 4$.

Solution

Start with the operation in the brackets.

$4 *(2 * y) = 4 *(2^2 - y)$

$= 4 *(4 - y)$

$= 4^2 - (4 - y)$

$= 16 - 4 + y$

Therefore, $16 - 4 + y = 4$

$12 + y = 4$

$y = -8.$

Example 2.3

Given that $x * y = 4x + 6y$, find $6 * 4$.

Solution

$6 * 4 = (4 \times 6) + (6 \times 4)$

$= 24 + 24$

$= 48$

Therefore, $6 * 4 = 48$.

Example 2.4

Given that $p * q = p^2 - q^2$, find $7 * 3$.

Solution

$7 * 3 = 7^2 - 3^2$

$= 49 - 9$

$= 40$

Therefore, $7 * 3 = 40$

Example 2.5

If $a * b = 3a + b$, find $5 * 8$.

Solution

$a * b = 3a + b$

$5 * 8 = (3 \times 5) + 8$

$= 15 + 8$

$= 23$

Therefore, $5 * 8 = 23$

Example 2.6

If $x * y = x + 3y$, find $5 * (8 * 6)$.

Solution

Start with operation in the brackets.

$5 * (8 * 6) = 5 * (8 + 3 \times 6)$

$= 5 * 26$

$= 5 + 3 \times 26$

$= 5 + 78$

$= 83$

Therefore, $5 * (8 * 6) = 83$.

Example 2.7

Given that $p \ \Delta \ q = 2p + 5q$, find $6 \ \Delta \ 3$.

Solution

$6 \Delta 3 = (2 \times 6) + (5 \times 3)$
$= 12 + 15$
$= 27$

Therefore, $6 \ \Delta \ 3 = 27$.

Example 2.8

Given that $m * n = \frac{m-n}{m+n}$, find $(3*2)$.

Solution

$3*2 = \frac{3-2}{3+2} = \frac{1}{5}$

Therefore, $(3*2) = \frac{1}{5}$.

Exercise 2.1

Answer the following questions:

1. Given that $*$ and Δ are two binary operations defined as $a * (b \Delta c) = a(b-c)$. If $a = 2$, $b = 5$ and $c = 4$, evaluate $(a * b) \Delta (a * c)$.

2. Given that $a * b = 5(a + 2b)$, find $4 * 2$.

3. Evaluate each of the following operations:
   (a) If $x * y = 3x + 6y$, find $2*(3*4)$.
   (b) If $p*q = 3p - 2q$, find $5*(4*3)$.
   (c) If $c \Delta d = c^2 - d^2$, find $9 \Delta 5$.
   (d) If $m*n = mn - \frac{n}{m}$, find $(3*6)*6$.

4. If $x * y$ is the operation "x cubed plus y", then find the value of $4*(3*2)$.

5. If $a*b = 3a^3 + 2b$, find the value of $(3*2)*6$.

6. The operation on integers d and k is defined by $d \Delta k = dk + 5d - 3k$, find the value of $3*2$.

7. Given that, $u\Delta v = \frac{\sqrt{uv}}{\sqrt{v}}$, find the value of $4\Delta 5$.

8. If $(a*b) = a^2 + b$, find y given that $4*(2*y) = 25$.

Brackets in computations

Brackets are symbols used to group things together. If you want to remove brackets in an operation, quantities inside the brackets are operated first followed by other operations.

For example, $x+(y+z)$ means that y and z are to be added together and their sum added to x. Similarly, $a\times(y+z)$ written in short as $a(y+z)$ means add together y and z, and then the sum is multiplied by a.

In expressions where there are mixtures of operations, the order of performing the operations is as follows:

1. Brackets (**B**) are first opened (**O**) followed by:
2. Division (**D**)
3. Multiplication (**M**)
4. Addition (**A**)
5. Subtraction (**S**)

This order of operations can be written in a short form as **BODMAS** so as to make it much easier to remember.

Example 2.9

Simplify the expression $4+3a\div(6a-3a)-3$.

Solution

$$
\begin{aligned}
4+3a\div(6a-3a)-3 &= 4+3a\div 3a-3 && \text{(by subtracting terms in brackets)}\\
&= 4+\frac{3a}{3a}-3 && \text{(by performing division)}\\
&= 4+1-3 && \text{(by performing addition)}\\
&= 5-3 && \text{(by performing subtraction)}\\
&= 2
\end{aligned}
$$

Therefore, $4+3a\div(6a-3a)-3=2$.

Sometimes, it is necessary to open brackets before proceeding with other operations.

Example 2.10

Simplify the expression $5x+(2x-3y)$.

Solution

$$
\begin{aligned}
5x+(2x-3y) &= 5x+2x-3y && \text{(by opening brackets)}\\
&= 7x-3y && \text{(by performing addition)}
\end{aligned}
$$

Therefore, $5x+(2x-3y)=7x-3y$.

Example 2.11

Simplify the expression $3y-(y-3x)$.

Solution

$$
\begin{aligned}
3y-(y-3x) &= 3y-1(y-3x)\\
&= 3y-y+3x && \text{(by opening brackets)}\\
&= 2y+3x && \text{(by performing subtraction)}
\end{aligned}
$$

Therefore, $3y-(y-3x)=2y+3x$.

Note that, every term inside the brackets is multiplied by the factor that is outside the brackets. Usually, a multiplication sign between a factor and brackets is omitted.

In general, when opening brackets where the sign before the bracket is negative, a term in the brackets will either change to positive if it is negative, or to negative if it is positive. However, when the sign before the brackets is positive, the signs in the brackets remain unchanged. The sign which comes after a bracket has no effect on the signs of terms in the brackets.

Exercise 2.2

Simplify the following expressions.

1. $m+n-(m-n)$
2. $a+2b-4(a+2b)\div(a+2b)$
3. $10p-4(2p+3q)$
4. $6y+(7a-2y)$
5. $5-4m+2m+2$
6. $(8z+3y-5z)-(3z+y)$
7. $m(2p-q)-m(3p+2q)-5(mp-mq)$
8. $4(m+n)-3(2m-n)+5(m-2n)$
9. $3t-2(4t-6)+2$
10. $(a+b)\times 4a\div 2a$
11. $2(m+2)+1$
12. $3(x+y)+x$
13. $3(a+b)+a+b$
14. $5(3x+6y)+2(x+y)$
15. $5(x+5)+(x-2)$
16. $2(p-q)+2(2p+3)$

Identities

An equation which holds true for all values of its variables is called an identity.

An equation is an identity if the LHS of the equation is equal to the RHS for all values of the variable involved. To determine whether an equation is an identity or not, substitute more than one value of the variable. If each side of the equation has the same value for at least two substituted values then the given equation is an identity.

For example, $2(z+1)=2z+2$ is an identity. If any value of z is substituted, each side has the same value. Thus, when $z=0$, the Left Hand Side (LHS) becomes $2(0+1)=2\times 1=2$ and the Right Hand Side (RHS) becomes $2\times 0+2=0+2=2$. Also, when $z=5$, LHS becomes $2(5+1)=2\times 6=12$ and RHS becomes $2\times 5+2=10+2=12$.

Alternatively, to determine whether an equation is an identity or not, show that the expression on one side of the equation is identical to the expression on the other side. For example, in $2(z+1)=2z+2$, the LHS becomes $2z+2$ when the brackets are opened. Therefore, $2(z+1)=2z+2$ is an identity.

Example 2.12

Determine whether or not $3x - 2 = 4x - 3$ is an identity.

Solution

If $x = 1$, the LHS becomes $3 \times 1 - 2 = 3 - 2 = 1$, and the RHS becomes $4 \times 1 - 3 = 1$.
The LHS is equal to the RHS when $x = 1$,
if $x = 0$, the LHS becomes $3 \times 0 - 2 = 0 - 2 = -2$ and the RHS becomes $4 \times 0 - 3 = 0 - 3 = -3$. The LHS is not equal to the RHS when $x = 0$
Therefore, $3x - 2 = 4x - 3$ is not an identity.

Example 2.13

Determine whether or not $x^2 - 1 = (x-1)(x+1)$ is an identity.

Solution

If $x = 4$, the LHS becomes $4^2 - 1 = 16 - 1 = 15$ and the RHS becomes $(4-1)(4+1) = 3 \times 5 = 15$.
RHS is equal to the LHS when $x = 4$.
Therefore, $x^2 - 1 = (x-1)(x+1)$ is an identity.

Example 2.14

Determine whether or not $x + 3 = 5$ is an identity.

Solution

Note that $x = 2$ is a solution to this equation. It is not advisable to use it for testing whether the equation is an identity or not because it is the only value that will satisfy it.
So, if $x = 3$, the LHS becomes $3 + 3 = 6$ while RHS is 5.
Since the LHS and RHS are not equal when $x = 3$.
Therefore, $x + 3 = 5$ is not an identity.

Exercise 2.3

Determine which of the following are identities:

1. $2y + 1 = 2(y+1)$
2. $3(a+2) = 2a + 6$
3. $2(r+1) = 2r + 2$
4. $2(a-b+3) = 2a - 2b + 6$

5. $2(p-1)+3=2p+1$
6. $4ad-6ac=a(4d-6c)$
7. $3pq+3qr=6pqr$
8. $\frac{3}{4}ac-\frac{3}{4}ac=\frac{3}{4}ac$
9. $\frac{1}{a}-\frac{1}{b}=\frac{2}{a+b}$
10. $\frac{-a+b}{a-b}=\frac{a-b}{b-a}$

Quadratic expressions

An expression whose highest exponent of the variable is 2 is called a **quadratic expression**. The following are examples of quadratic expressions:

(i) $4z^2+3$, the highest exponent of z is 2.

(ii) $6y^2+y$, the highest exponent of y is 2.

(iii) $3n^2-2n+1$, the highest exponent of n is 2.

In general, a quadratic expression is written in the form ax^2+bx+c, where $a\neq 0$ and a, b, c are real numbers. If $a=0$, the expression becomes linear. From the expression ax^2+bx+c, the term bx is called the **middle term**, 'a' is called the coefficient of x^2, 'b' is called the coefficient of x, and 'c' is called the constant term.

Thus, in the quadratic expression $4x^2-6x+7$, the coefficient of x^2 is $a=4$, the coefficient of x is $b=-6$ and the constant $c=7$.

Product of two linear expressions

Quadratic expressions may arise from the product of two linear expressions. For example in Figure 2.1, if the width of a rectangle is $(y+1)$ unit and its length is $(2y+3)$ unit, then the area is $(2y+3)(y+1)$ square units.

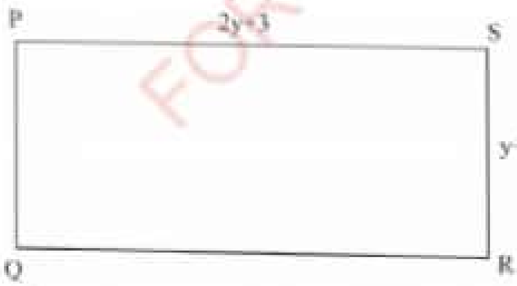


Figure 2.1: *Rectangle PQRS*

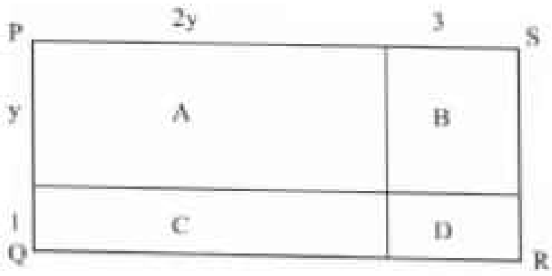


Figure 2.2: *Regions of rectangle PQRS*

Consider Figure 2.1, a rectangle PQRS whose sides are $(2y + 3)$ units and $(y + 1)$ units respectively.

When rectangle PQRS is subdivided into four rectangles A, B, C and D, as shown in Figure 2.2, the following is true:

Area of region A $= 2y \times y = 2y^2$ square units.

Area of region B $= 3 \times y = 3y$ square units.

Area of region C $= 1 \times 2y = 2y$ square units.

Area of region D $= 3 \times 1 = 3$ square units.

The total Area of regions A, B, C, and D $= (2y^2 + 3y + 2y + 3)$ square units.

$= (2y^2 + 5y + 3)$ square units.

But the total area is $(2y+3)(y+1)$ square units.

Therefore, $(2y+3)(y+1) = 2y^2 + 5y + 3$.

The expression $2y^2 + 5y + 3$ is also called the expanded form of $(2y+3)(y+1)$. Each term in the first pair of brackets is multiplied by each term in the second pair of brackets. The following are two alternative ways of multiplying two linear expressions:

Alternative 1

$$\begin{array}{r} 2y+3 \\ \times \quad y+1 \\ \hline 2y^2+3y \quad \\ + \quad 2y+3 \\ \hline 2y^2+5y+3 \end{array}$$

Therefore, $(2y+3)(y+1) = 2y^2 + 5y + 3$.

Alternative 2

$$\begin{aligned}(2y+3)(y+1) &= 2y(y+1) + 3(y+1) \\ &= 2y^2 + 2y + 3y + 3 \\ &= 2y^2 + 5y + 3\end{aligned}$$

Thus, $2y^2 + 5y + 3$ is the expanded form of $(2y+3)(y+1)$.

Therefore, $(2y+3)(y+1) = 2y^2 + 5y + 3$.

Example 2.15

Expand $(z+2)(2z-3)$.

Solution

$$(z+2)(2z-3) = z(2z-3)+2(2z-3)$$
$$= 2z^2-3z+4z-6$$
$$= 2z^2+z-6$$

Therefore, $(z+2)(2z-3) = 2z^2+z-6$.

Example 2.16

Expand $(6x+5)^2$

Solution

$$(6x+5)^2 = (6x+5)(6x+5)$$
$$= 6x(6x+5)+5(6x+5)$$
$$= 36x^2+30x+30x+25$$
$$= 36x^2+60x+25$$

Therefore, $(6x+5)^2 = 36x^2+60x+25$.

Example 2.17

Find the coefficient of n and n^2 in the expansion of $(n+9)(n+3)$.

Solution

$$(n+9)(n+3) = n(n+3)+9(n+3)$$
$$= n^2+3n+9n+27$$
$$= n^2+12n+27$$

Therefore, the coefficients of n and n^2 are 12 and 1, respectively.

Note that, from the general quadratic expression ax^2+bx+c, where a, b and c are real numbers and $a \neq 0$;

(a) If $a=0$, the expression becomes linear. For example, $6x+4$, in this case, $a=0$, $b=6$ and $c=4$.

(b) If $b=0$, the expression is a quadratic but without the middle term. For example, $4x^2+8$, in this case, $a=4$, $b=0$ and $c=8$.

(c) If $c=0$, the expression is a quadratic but without the constant term. For example, $10x^2-2x$, in this case, $a=10$, $b=-2$ and $c=0$.

(d) The basic identities in quadratic expressions are the following:

(i) $(a+b)^2=a^2+2ab+b^2$

(ii) $a^2-b^2=(a+b)(a-b)$

(iii) $(x+a)(x+b)=x^2+(a+b)x+ab$

Exercise 2.4

Answer the following questions:

In questions 1 to 12, write the expanded form of each of the given expressions:

1. $(2x+3)(x+1)$
2. $(4x-3)(2x+1)$
3. $y(y+6)$
4. $(3y+4)^2$
5. $(7n-2)^2$
6. $(a+b)^2$
7. $(a-b)^2$
8. $(4x-3)(2x+1)$
9. $(3p+2q)(6p-2q)$
10. $(x+y)(x-y)$
11. $(x+y)(x-y)$
12. $(-3+y)(y+4)$

13. Find the area of a triangle whose base is $(6x + 1)$ cm and height is $(4x - 2)$ cm. Write the answer in the expanded form.

14. Find the amount of money required to buy $7y$ items if each item costs $(6y + 5)$ shillings. Give the answer in the expanded form.

15. Given three consecutive whole numbers, write the expanded form of the product of the first and the third if the second number is n.

Factorization of algebraic expressions

When expanding an expression, the resulting expression has no brackets. The reverse operation is called **factorization** which gives an expression that contains brackets. When factorizing an expression, the factor which is common in all the terms is identified and written outside of a pair of brackets (factored out).

Consider the expression $8a + 8b$. The number 8 is multiplied by both letters a and b. So, 8 is common to both terms. Therefore, it can be taken outside of brackets and the expression can be written as $8a+8b=8(a+b)$.

Sometimes, more than one number or letter can be taken outside the brackets. For example, $5xy+10xz=5x(y+2z)$.

Example 2.18

Factorize the expression $cx+cy$.

Solution

If c is the common factor, it has to be taken outside a pair of brackets as follows:
$cx+cy=c(x+y)$.

Observe that, expansion of the right hand side results to $cx+cy$.

Example 2.19

Factorize $3xy-9xz$ completely.

Solution

$3xy-9xz = 3xy-3\times 3xz$ (3 and x are common factors)

$= 3x(y-3z)$

Therefore, $3xy-9xz = 3x(y-3z)$.

Example 2.20

Factorize mn^2-8nm^2.

Solution

Since the common factor is mn, then

$mn^2-8nm^2 = m\times n\times n-8\times n\times m\times m$

$= mn(n-8m)$

Therefore, $mn^2-8nm^2 = mn(n-8m)$.

Exercise 2.5

Factorize the following expressions:

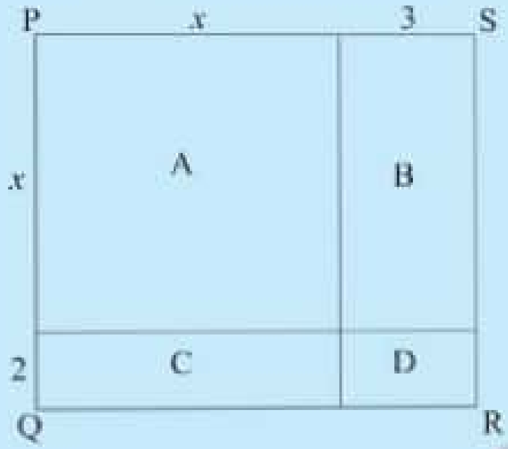
1. $5a+5b$
2. $7a-7y$
3. $2m+2n$
4. $am-an$
5. $bx+by$
6. $ax+ay+az$
7. $3x+6y$
8. $7m-21n$
9. $15p-5q$
10. $100w+1000$
11. $2rs+4s$
12. $pqr-3pr$
13. $25kl-35kt$
14. $abc+bcd+cde$
15. $6pq-2pt+8pm$
16. $\frac{1}{2}nm+\frac{1}{6}nt$
17. $4xyz+16xyn+8xyr$
18. $0.3ab+0.5cb$
19. $\frac{1}{2}nm+\frac{1}{3}nt-\frac{1}{4}nx$
20. $18pqr-24pwr$
21. $0.125yu-0.5u$

Factorization of quadratic expressions

Activity: Recognizing the area of a rectangular region

Perform the following tasks in groups:

1. Draw the following figure PQRS on a plane paper.



2. Calculate the area of regions A, B, C and D separately.
3. Find the sum of areas obtained in step 2.
4. Write the expressions for length and width of figure PQRS.
5. Calculate the area of figure PQRS.
6. Compare the answers obtained in step 3 and 5. What comment can you make about the product of $(x+3)$ and $(x+2)$?

From the activity, the expression $(x+3)(x+2)$ was expanded to obtain x^2+5x+6. Hence, $(x+3)$ and $(x+2)$ are factors of x^2+5x+6.

The following are methods used to factorise quadratic expressions:

(i) Splitting the middle term

(ii) Inspection

(iii) Difference of two squares

(iv) Perfect squares

Factorization by splitting the middle term

You have been factorizing expressions which have common factors. Suppose you want to factorize an expression like $ax^2 + bx + c$. The middle term (bx) can be splitted into two terms. But not all choices of splitting the middle term lead to the required factors. A quick method of picking the correct choice is to observe the coefficients and the constant term as follows:

(i) Consider the coefficients a, b and the constant term c,

(ii) Find the two numbers whose sum is b and whose product is ac,

(iii) This can be easily done by finding the factors of ac in pairs and find the pair whose sum is b.

Example 2.21

Factorize the expression $2x^2 + 7x + 6$ by splitting the middle term.

Solution

The coefficients are 2 and 7 and the constant term is 6.

$a = 2$, $b = 7$ and $c = 6$. So $ac = 2 \times 6 = 12$.

The pairs of factors of 12 are 1 and 12, 2 and 6, 3 and 4.

The sum of 1 and 12 is 13

The sum of 2 and 6 is 8

The sum of 3 and 4 is 7

Therefore, the correct choice is 3 and 4. So the terms are $3x$ and $4x$.

$$\begin{aligned} 2x^2 + 7x + 6 &= 2x^2 + 4x + 3x + 6 \\ &= (2x^2 + 4x) + (3x + 6) \\ &= 2x(x + 2) + 3(x + 2) \\ &= (x + 2)(2x + 3) \end{aligned}$$

Therefore, $2x^2 + 7x + 6 = (x + 2)(2x + 3)$.

Example 2.22

Factorize $6x^2 - 11x + 4$ by splitting the middle term.

Solution

Required to find the factors of 24 whose sum is -11.

The pairs of factors of 24 are: -1 and -24, -2 and -12, -3 and -8, -6 and -4.

Therefore, the correct choice is -3 and -8 since $-3 + -8 = -11$ and $(-3) \times (-8) = 24$.

Thus, $-11x = -3x - 8x$.

$$\begin{aligned}\text{So that, } 6x^2 - 11x + 4 &= 6x^2 - 3x - 8x + 4\\ &= (6x^2 - 3x) - (8x - 4)\\ &= 3x(2x - 1) - 4(2x - 1)\\ &= (2x - 1)(3x - 4)\end{aligned}$$

Therefore, $6x^2 - 11x + 4 = (2x - 1)(3x - 4)$.

Example 2.23

Factorize $2x^2 + x - 10$ by splitting the middle term.

Solution

The correct choice of factors of -20 is -4 and 5; Thus, $x = -4x + 5x$.

$$\begin{aligned}\text{So that, } 2x^2 + x - 10 &= 2x^2 - 4x + 5x - 10\\ &= (2x^2 - 4x) + (5x - 10)\\ &= 2x(x - 2) + 5(x - 2)\\ &= (x - 2)(2x + 5)\end{aligned}$$

Therefore, $2x^2 + x - 10 = (x - 2)(2x + 5)$.

Math F2 – (BOOK) (DUMMY) Sept 4.indb 65

Example 2.24

Factorize x^2+6x+9 by splitting the middle term.

Solution

The correct choice of factors of 9 is 3 and 3. Thus, $6x=3x+3x$.

So that, $x^2+6x+9=x^2+3x+3x+9$

$$\begin{aligned}&=(x^2+3x)+(3x+9)\\&=x(x+3)+3(x+3)\\&=(x+3)(x+3)\\&=(x+3)^2\end{aligned}$$

Therefore, $x^2+6x+9=(x+3)^2$.

In this example, the quadratic expression has two factors which are identical. A quadratic expression in which the two factors are identical is called a **perfect square.** The general form of a perfect square is the identity $(a+b)^2=a^2+2ab+b^2$.

Exercise 2.6

Factorize the following expressions by splitting the middle term:

1. x^2+3x+2
2. $6y^2+11y+4$
3. $2x^2-17x+8$
4. y^2-7y+6
5. t^2+6t+8
6. y^2-3y+2
7. $3d^2-2d-8$
8. $6x^2+5x-6$
9. $m^2+11m+10$
10. $t^2-17t+60$
11. $3-5a+2a^2$
12. $s^2+3s-28$
13. $x^2+10x+21$
14. $y^2+5y-36$
15. $2x^2-7x-15$
16. $c^2-18c+45$
17. $12x^2+27x-39$
18. $x^2-7xy+12y^2$
19. $3y^2-11y+10$
20. $4t^2+5t+14$
21. $2x^2+x-1$

Factorization by inspection

Another method of factorizing a quadratic expression is by inspection. For example, to factorize $m^2 + 6m + 8$ by inspection, we need to fill in the brackets of the statement $m^2 + 6m + 8 = (\ )(\ )$.

The first term in the given expression is m^2. This can be obtained by putting m first in each pair of the brackets. The next term to consider is the last term in the given expression, which is 8. This number is the product of the last terms in the two pairs of brackets and the correct choice is from the following pairs: 8 and 1, –8 and –1, 4 and 2, –4 and –2. The possibilities to consider are as follows:

(a) $(m+8)(m+1)=m^2+9m+8$

(b) $(m-8)(m-1)=m^2-9m+8$

(c) $(m+4)(m+2)=m^2+6m+8$

(d) $(m-4)(m-2)=m^2-6m+8$

Note that, the coefficient of m on the RHS is the sum of the numbers appearing in each pair of brackets. That is, $9=8+1$; $-9=(-8)+(-1)$ and so on.
Hence, (a), (b) and (d) can be discarded at once.
The required result is $m^2 + 6m + 8 = (m+4)(m+2)$

Example 2.25

Factorize $a^2 - 11a + 10$ by inspection.

Solution

Write the expression $a^2 - 11a + 10 = (\ )$
The first term in each pair of brackets is a
The constant term is 10
The middle term is $-11a$
Pairs of factors of 10 are 5 and 2; – 5 and – 2; 10 and 1; – 10 and –1.
From these pairs, only – 10 and –1 add up to – 11
Therefore, $a^2 - 11a + 10 = (a-10)(a-1)$.

Example 2.26

Factorize $x^2+9x+18$ by inspection.

Solution

The factors of 18 that add up to 9 are 3 and 6.

Therefore, $x^2+9x+18=(x+3)(x+6)$.

Example 2.27

Factorize $y^2-4y-12$ by inspection.

Solution

The coefficient of y is -4 and the constant term is -12. The two factors of -12 whose sum is -4 are -6 and 2.

Therefore, $y^2-4y-12=(y-6)(y+2)$.

Example 2.28

Factorize $m^2+2m-15$ by inspection.

Solution

The coefficient of m is 2 and the constant term is -15.

The factors of -15 whose sum is 2 are -3 and 5.

Therefore, $m^2+2m-15=(m-3)(m+5)$.

Factorization by difference of two squares

An expression of the form a^2-b^2 is called the difference of two squares. It originated from a product of two factors $(a+b)$ and $(a-b)$, that is, $(a+b)(a-b)=a^2-b^2$.

Example 2.29

Factorize $a^2 - 25b^2$.

Solution

Write $a^2 - 25b^2$ as a difference of two squares.

$a^2 - 25b^2 = a^2 - 5^2b^2$ (writing 25 as a square number)

$a^2 - 25b^2 = a^2 - (5b)^2$ (writing 5^2b^2 as a single base)

$= (a - 5b)(a + 5b)$

Therefore, $a^2 - 25b^2 = (a - 5b)(a + 5b)$.

Example 2.30

Factorize $x^2 - 81$.

Solution

Write $x^2 - 81$ as a difference of two squares.

$x^2 - 81 = x^2 - 9^2$

$= (x + 9)(x - 9)$

Therefore, $x^2 - 81 = (x + 9)(x - 9)$.

Example 2.31

Find the exact value of $50\ 001^2 - 49\ 999^2$.

Solution

$50\ 001^2 - 49\ 999^2 = (50\ 001 + 49\ 999)(50\ 001 - 49\ 999)$

$= 100\ 000(2)$

$= 200\ 000$

Therefore, $50\ 001^2 - 49\ 999^2 = 200\ 000$.

Factorization by perfect squares

It has been verified that;

$a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$

These two quadratic identities $(a+b)^2$ and $(a-b)^2$ are called perfect squares.

Example 2.32

Factorize $x^2 - 6x + 9$ in the form of $(x+a)^2$.

Solution

$$x^2 - 6x + 9 = (x-3)(x-3)$$
$$= (x-3)^2$$

Therefore, $x^2 - 6x + 9 = (x-3)^2$.

Example 2.33

Factorize $a^2 + \frac{6}{5}a + \frac{9}{25}$.

Solution

Write $a^2 + \frac{6}{5}a + \frac{9}{25} = (a + ...)^2$ (the first term in the bracket must be a)

The second term in the bracket is obtained by taking the **square root of the constant term.**

That is, $\sqrt{\frac{9}{25}} = \frac{3}{5}$.

Therefore, $a^2 + \frac{6}{5}a + \frac{9}{25} = \left(a + \frac{3}{5}\right)^2$.

Note:

1. In $a^2+2ab+b^2=(a+b)^2$, the square of the sum of two quantities is equal to the sum of their squares plus twice their product.
2. In $a^2-2ab+b^2=(a-b)^2$, the square of the difference of two quantities is equal to the sum of their squares minus twice their product.
3. In all perfect squares, the constant term is the square of half the coefficient of the linear (middle) term.

Exercise 2.7

Answer the following questions:

Factorize the following quadratic expressions by inspection:

1. x^2+6x+8
2. x^2+3x+2
3. $2y^2+9y+7$
4. $4u^2+9u+5$
5. $12m^2+37m+21$
6. $22a^2+25a+7$

Factorize the following quadratic expressions by perfect square:

7. $c^2+\frac{8}{9}c+\frac{16}{81}$
8. $g^2+22g+121$
9. y^2-4y+4
10. $k^2+34k+289$
11. $w^2-16w+64$
12. $4r^2-12r+9$
13. $9x^2-6x+1$
14. $4+9a^2-12a$

Factorize the following quadratic expressions by using the method of difference of two squares.

15. $9x^2-121$
16. x^2-100
17. $4x^2-y^2$
18. $64a^2-25$
19. $1-a^2b^2$
20. $121a^2-64a^2$

21. Simplify $(a+b)^2 - 3c(a+b)$.

22. Factorize $-6+3w^2-7w$ by splitting the middle term.

23. If $kx^2+24x+16$ is a perfect square, find the value of k.

24. Factorize x^2-y^2 and hence use the results to evaluate the value of

$-6+3w^2-7w$

25. Find the exact value of $1\,008^2 - 992^2$.

Chapter summary

1. A binary operation is an operation that applies to two quantities or expressions.
2. A binary operation may be denoted by symbols such as $*$ or Δ.
3. In computations with brackets, remember to simplify the expression inside the brackets first. Work with the innermost pair, moving outwards.
4. Calculations inside brackets (parenthesis) are done first. When you have more than one set of brackets, do the inner brackets first.
5. An expression whose highest exponent of the variable is 2 is called a quadratic expression. For example, $3n^2+2n-1$ is a quadratic expression.
6. A quadratic expression is written in the form ax^2+bx+c where a, b and c are real numbers and $a \neq 0$.
7. To factorize a quadratic expression is the same as writing it as a product of its factors.
8. Methods used in factorizing quadratic expressions include splitting the middle term, inspection, difference of two squares and perfect squares.

Revision exercise 2

Answer the following questions:

1. To the sum of $(2a-3b-4c)$ and $(4b-3a-2c)$, add the sum of $(3c-4b-5a)$ and $(-a+2b-5c)$ and the sum of $(2a-3b-4c)$ and $(4b-3a-2c)$.
2. In a train which has 3 class cabins, there are $3x^2-13x+4$ passengers. Among them, $x-y+z$ are in the first class cabins and $2x-3y-z$ are in the third class cabins.
   (a) Find an algebraic expression for the number of passengers in the second class cabins.
   (b) What is the number of passengers in each class cabin if $x=43$, $y=38$ and $z=6$?
3. Simplify the following expressions:
   (a) $a(a-b)-b(b+a)$ (b) $x-(x-y)+(2x-7)$
4. Factorize fully by grouping terms with common factors:
   (a) $2ax+3ay-8bx-12by$ (b) $3ac+2ba+ad+6bc$
   (c) $6pr+6qs-9ps-4qr$ (d) $8x^2+6xy-4xy-3y^2$
   (e) $r-1-r^2+r$
5. Expand the following expressions:
   (a) $(a-4)(a+6)$ (b) $(2x-3)(3x-3)$
   (c) $(x+2y)(2p+q)$ (d) $(2a+3b)(3a+2b)$
   (e) $(3r-3s)(r-4s)$
6. Find the coefficient of x in the expansion of $(4x-3)(x+2)$.
7. Find the constant term in the expansion of $(a-3)(6a-2)$.
8. Find the coefficient of y^2 in the expansion of $(4y-6)(2y+3)$.
9. The lengths of the parallel sides of a trapezium are $(x+4)$ and $(x+2)$. The perpendicular distance between the parallel sides is $(2x-3)$. Find the area of the trapezium and write the answer in the form ax^2+bx+c.
10. Simplify $(4a-6)(2a+5)-(2a+5)(4a-3)$.
11. A piece of wire y cm long is bent to form a square. Find the area of the square.
12. Simplify the algebraic expression $a^2+6a+a+8-6\times a^2$.

13. Determine whether or not each of the following is an identity:

(a) $\frac{1-b}{1+b}=\frac{b-1}{b+1}$

(b) $x(y+2)+y(x+2)=2(xy+x+y)$

(c) $\frac{ad+cx}{ad}=cx$

(d) $(a+b)^2=a^2+b^2$

(e) $a(a+1)=a(1-a)$

(f) $2(x-1)-4(2-x)=2(3x-5)$

(g) $x(y+z)+x=xy+x+xz$

14. Give the missing term so that each of the following expressions becomes an identity:

(a) $6x-7y-8a+9b=(6x-7y)-(\ )$

(b) $2p-3q-4r-5s=(2p-3q)-(\ )$

(c) $6x+6-9x-4=(x+1)-(\ )$

(d) $6ab+2a-9bc+3c=2a(3b+1)+(\ )$

(e) $(3a-4b)(2c-3d)=(6ac-9ad)+12bd-(\ )$

15. Factorize and simplify each of the following expressions where possible:

(a) $9a^2-25b^2$ (b) $(2c+3)^2-c^2$ (c) $36(x+2y)^2-25(2x-y)^2$

16. Verify that a^2-b^2 is not equal to $(a-b)^2$ by substituting $a=2$ and $b=1$.

17. Find the exact values of the following:

(a) $23\ 756^2-23\ 754^2$ (b) 672^2-328^2

18. Find the factors of $15t^2-14t-8$.

19. Which of the following expressions are perfect squares:

(a) x^2+3x+3 (b) x^2-2x-1

(c) x^2-2x+1 (d) $4x^2+20x+25$

Chapter Three

Quadratic equations

Introduction

A quadratic equation is any algebraic equation whose highest exponent of a variable is two. The word quadratic comes from the Latin word "quadratus" which means square. In this chapter you will learn to solve quadratic equations by factorization, splitting the middle term, and by completing the square. You will also learn to solve equations in the form of difference of two squares, and perfect squares. Competences developed in this chapter will enable you to calculate business profits, surface areas like floor of the room, piece of land and surfaces of rectangular objects. You will also be able to determine the speed of moving objects like cars and planes. Furthermore, you will be able to appreciate the use of quadratic expression in the manufacture of satellite dishes, car head lamps and other objects with parabolic shapes.

Solving quadratic equations

Let, ax^2+bx+c be a quadratic expression where x is an unknown variable, a and b are real numbers ($a \neq 0$) called coefficients, and c is a real number called a constant. When the expression is equal to zero, it becomes a quadratic equation. So, $ax^2+bx+c=0$ is a **quadratic equation**, where a is the coefficient of x^2, b is the coefficient of x, and c is a constant term. For example, in the quadratic equation $8x^2+6x+2=0$,

8 is the coefficient of x^2,

6 is the coefficient of x and,

2 is the constant term.

Similarly, for the equation $2x^2-32=0$, $a=2$, $b=0$ and $c=-32$ and for the equation $4x^2-16x=0$, $a=4$, $b=-16$ and $c=0$.

Note that;

(i) $3x^2 = -7x - 1$ can be written in the form of $ax^2 + bx + c = 0$ as $3x^2 + 7x + 1 = 0$, where $a = 3$, $b = 7$ and $c = 1$.
Also, $7x = 4(x^2 + 1)$ can be written as $4x^2 - 7x + 4 = 0$.

(ii) When a quadratic equation is factorized, then either one of the factors or both factors must be equal to 0. In general, $(x + a)(x + b) = 0$, if $x + a = 0$ or $x + b = 0$ or both are zero. This is called the **zero factor theorem.**

Therefore, if the product of two algebraic factors is 0, then one of the factors (or possibly both of them) must be 0. This concept is used to solve quadratic equations given as a product of factors.

Example 3.1

Solve the equation $(x + 4)(x - 3) = 0$.

Solution

If $(x + 4)(x - 3) = 0$, then either $x + 4 = 0$ or $x - 3 = 0$.

Therefore, $x = -4$ and $x = 3$.

Example 3.2

Solve the equation $a(a + 3) = 0$.

Solution

If $a(a + 3) = 0$ then, either $a = 0$ or $a + 3 = 0$

Therefore, $a = 0$ and $a = -3$.

Solving quadratic equations by factorization

The following are steps for solving quadratic equations by factorization:

1. Write the given quadratic equation in standard form $ax^2+bx+c=0$.
2. Factorize $ax^2+bx+c=0$ into two linear factors.
3. Equate each linear factor to zero (zero product rule).
4. Solve these linear equations to obtain the two roots of the given quadratic equation.

Solving quadratic equations with no constant term

In general, a quadratic equation of the form $ax^2+bx=0$, where $a\neq 0$ can be solved as follows:

$ax^2+bx=0$.

$x(ax+b)=0$.

Either, $x=0$ or $ax+b=0$.

Then, $x=0$ or $ax=-b$.

Therefore, $x=0$ and $x=-\frac{b}{a}$.

Consider a quadratic equation $x^2-6x=0$. This can be written as $x(x-6)=0$ which means that, the product of x and $x-6$ is zero. Given $x(x-6)=0$, then, either $x=0$ or $x-6=0$. Solving $x-6=0$, gives $x=6$. Hence, the solution is $x=0$ and $x=6$.

Example 3.2

Solve the quadratic equation $3x^2-x=0$.

Solution

$3x^2-x=0$.

$x(3x-1)=0$

either, $x=0$ or $3x-1=0$.

Therefore, $x=0$ and $x=\frac{1}{3}$.

Example 3.3

Solve the quadratic equation $x^2+8x=0$.

Solution

$x^2+8x=0$

$x(x+8)=0$

either, $x=0$ or $x+8=0$

Therefore, $x=0$ and $x=-8$.

Exercise 3.1

Solve the following quadratic equations:

1. $x(x-5)=0$
2. $x^2+7x=0$
3. $h(h-3)=0$
4. $5k^2=2k.$
5. $4f^2-11f=0$
6. $3p^2+5p=0$
7. $5x^2=7x.$
8. $7x^2=4x^2+6x$
9. $t^2-4t=0$
10. $7x-x^2=x^2+3x$
11. $x(3x-7)=0$
12. $3x^2+4x=0$
13. $3x=5x^2$
14. $x^2-5x=0$
15. $2x(x-7)=0$
16. $x(x+2)-9(x+2)=0$
17. $2p=2p^2$
18. $x^2=6x$
19. $x^2-16x=0$
20. $14x^2+6x=0$

Solving quadratic equations by splitting the middle term

The middle term of a quadratic expression may have a positive or negative coefficient. To split the middle term is to find two terms whose sum is equal to the middle term, and whose product is the same as the product of the coefficient of x^2 and the constant term.

The following are useful steps for solving quadratic equations by splitting the middle term:

1. Write the given quadratic equation in standard form $ax^2+bx+c=0$.
2. Split the middle term by finding two terms whose sum is equal to the coefficient of the middle term and whose product is the same as the product of the coefficient of x^2 and the constant term.
3. Factorize the resulting equation in step 2 into two linear factors.
4. Equate each linear factor equal to zero.
5. Solve these linear equations to obtain the two roots of the given quadratic equation.

Solving quadratic equations with positive coefficients of the middle term

Let the quadratic equation $ax^2+bx+c=0$ have positive coefficients a and b. The equation can be factorized by splitting the middle term "bx" as shown in the following examples.

Example 3.4

Solve the quadratic equation $10x^2+9x+2=0$.

Solution

Comparing $10x^2+9x+2=0$ with $ax^2+bx+c=0$

$a=10,\ b=9,\ c=2$.

The product of $ac=10\times 2=20$.

Factors of $ac=20$ are 1, 2, 4, 5, 10, 20.

Among these factors, 4 and 5 give a product of 20 and sum of 9.

Now use 4 and 5 to split the middle term;

$10x^2+5x+4x+2=0$ (splitting the middle term "$9x=5x+4x$")

$5x(2x+1)+2(2x+1)=0$ (factorizing into two linear factors)

$(2x+1)(5x+2)=0$.

Either, $2x+1=0$ or $5x+2=0$ (applying the zero factor concept).

Therefore, $x=-\frac{1}{2}$ and $x=-\frac{2}{5}$.

Example 3.5

Solve fo t if $t^2+6t+8=0$.

Solution

$t^2+6t+8=0$.

$t^2+4t+2t+8=0$. (splitting the middle term)

$t(t+4)+2(t+4)=0$ (factorizing into two linear factors)

$(t+4)(t+2)=0$.

Either, $t+4=0$ or $t+2=0$ (applying the zero factor theorem).

Therefore, $t=-4$ and $t=-2$.

Exercise 3.2

Solve the following quadratic equations by splitting the middle term:

1. $x^2+7x+6=0$
2. $2x^2+11x+5=0$
3. $2x^2+9x+9=0$
4. $x^2+11x+30=0$
5. $x^2+10x+21=0$
6. $7x^2+22x+3=0$
7. $4x^2+12x+5=0$
8. $x^2+36x+35=0$
9. $2x^2+5x-12=0$
10. $3x^2+4x+1=0$.
11. $x^2=-2x-1$
12. $x^2+15=-8x$

Solving quadratic equations with negative coefficients in their middle terms

Quadratic equations with negative coefficients in the middle terms are solved in the same way as those with positive coefficients in the middle terms.

Example 3.6

Solve the quadratic equation $12x^2 - 22x + 8 = 0$.

Solution

Comparing $12x^2 - 22x + 8 = 0$ with $ax^2 + bx + c = 0$, results to:

$a = 12,\ b = -22,\ c = 8.$

$ac = 12 \times 8 = 96.$

The factors of 96 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96,

Thus 6 and 16 should be considered with negative signs,

That is -6 and -16 so as to give a sum of -22 and product of 96.

So $-22 = -6x + -16x$.

The quadratic equation $12x^2 - 22x + 8 = 0$ can be re-written as:

$12x^2 - 16x - 6x + 8 = 0.$ (splitting the middle term).

$(12x^2 - 16x) - (6x - 8) = 0,$

$4x(3x - 4) - 2(3x - 4) = 0$ (factorizing into two linear factors).

$(3x - 4)(4x - 2) = 0$

Either $3x - 4 = 0$ or $4x - 2 = 0$ (by the zero factor concept).

Therefore, $x = \frac{4}{3}$ and $x = \frac{1}{2}$.

Example 3.7

Solve the quadratic equation $2h^2 - 7h = 39$.

Solution

$2h^2 - 7h - 39 = 0$

$2h^2 + 6h - 13h - 39 = 0$ (splitting the middle term)

$2h(h+3) - 13(h+3) = 0$ (factorizing into two linear factors)

$(h+3)(2h-13) = 0$

Either, $h+3=0$ or $2h-13=0$ (by the zero factor concept)

Therefore, $h=-3$ and $h=\frac{13}{2}$.

Example 3.8

Solve the equation $y^2 - 2y - 24 = 0$.

Solution

$y^2 - 2y - 24 = 0$

$y^2 - 6y + 4y - 24 = 0$ (splitting the middle term)

$y(y-6) + 4(y-6) = 0$ (factorizing into two linear factors)

$(y-6)(y+4) = 0$ (by the zero factor concept)

Either, $y-6=0$ or $y+4=0$.

Therefore, $y=6$ and $y=-4$.

Example 3.9

Solve the following quadratic equation $6e^2 - 105 = 31e$.

Solution

$6e^2 - 31e - 105 = 0$

$6e^2 + 14e - 45e - 105 = 0$

$2e(3e + 7) - 15(3e + 7) = 0$

$(3e + 7)(2e - 15) = 0$

Either, $3e + 7 = 0$ or $2e - 15 = 0$

Therefore, $e = \frac{15}{2}$ and $e = -\frac{7}{3}$.

Exercise 3.3

Solve the following quadratic equations by factorization:

1. $x^2 - 10x + 9 = 0$
2. $x^2 - 5x + 4 = 0$
3. $3x^2 = 10x + 8$
4. $x^2 - x - 12 = 0$.
5. $2x^2 - 7x + 6 = 0$
6. $2u^2 + 17 = 19u$
7. $x^2 - 10x + 21 = 0$
8. $3x^2 - 5 = 14x$
9. $10x^2 - 3x - 1 = 0$
10. $6x^2 - 20x + 6 = 0$.
11. $3x^2 - 15x + 18 = 0$
12. $4x^2 - 13x + 3 = 0$
13. $4q^2 - 4q - 24 = 0$
14. $3x(3x - 2) + 1 = 0$
15. $n^2 - 7n - 120 = 0$
16. $20x^2 = 7x + 6$.
17. $2q^2 - 12q = 14$
18. $(2x + 1)^2 = (3x - 2)^2$
19. $12 + 2x = 2x^2$
20. $3v^2 + 12 = 13v$

Quadratic equations which lead to the difference of two squares

A quadratic equation of the form $x^2 - c^2 = 0$ (c is a constant) involves the difference of two squares. Such equations can be solved by the factorization method of the difference of two squares, that is $x^2 - c^2 = (x+c)(x-c) = 0$.

Example 3.10

Solve the quadratic equation $x^2 - 9 = 0$.

Solution

$x^2 - 9 = 0$

$x^2 - 3^2 = 0$

$(x+3)(x-3) = 0$

Either, $x + 3 = 0$ or $x - 3 = 0$

Therefore, $x = -3$ and $x = 3$.

Example 3.11

Solve the quadratic equation $9x^2 - 4 = 0$.

Solution

$9x^2 - 4 = 0$

$(3x)^2 - 2^2 = 0$

$(3x+2)(3x-2) = 0$

Either, $3x - 2 = 0$ or $3x + 2 = 0$.

Therefore, $x = -\frac{2}{3}$ and $x = \frac{2}{3}$.

Perfect squares

A quadratic equation is a perfect square if it has identical factors.

Consider the general equation $ax^2 + bx + c = 0$.

It implies that $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, where $a \neq 0$ (dividing by a through out)

Then, $\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$ (completing the square)

Upon re–writing the constant terms under one denominator;

$$\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right) = 0$$

This equation becomes a perfect square if $4ac - b^2 = 0$.

Thus, $\left(x + \frac{b}{2a}\right)^2 = 0$ is a perfect square since it has identical roots $x = \frac{-b}{2a}$.

Quadratic equations involving perfect squares can be solved by the method of **factorization of perfect squares.**

Example 3.12

Solve the quadratic equation $x^2 + 6x + 9 = 0$ by factorization of perfect squares.

Solution

For a perfect square $4ac - b^2 = 0$

check if $x^2 + 6x + 9 = 0$ is a perfect square

From the equation $a = 1$, $b = 6$, and $c = 9$.

$$\begin{aligned} 4ac - b^2 &= 4 \times 1 \times 9 - 6^2 \\ &= 36 - 36 \\ &= 0 \end{aligned}$$

Therefore, $x^2 + 6x + 9 = 0$ is a perfect square

Thus, $x^2 + 6x + 9 = \left(x + \frac{6}{2}\right)^2 = (x + 3)^2 = 0$

Either, $x + 3 = 0$ or $x + 3 = 0$

Thus, $x = -3$ or $x = -3$.

Therefore, $x = -3$ is a repeated root.

Alternatively, by comparing $x^2 + 6x + 9 = 0$ with $ax^2 + bx + c = 0$,

$a = 1,\ b = 6,\ c = 9$

For a perfect square; $\left(x + \frac{b}{2a}\right)^2 = 0$

$x^2 + 6x + 9 = 0$ is a perfect square

Thus $\left(x + \frac{6}{2 \times 1}\right)^2 = (x + 3)^2 = 0$

Therefore, $x = -3$ is a repeated root.

Example 3.13

Solve the quadratic equation $x^2 - 14x + 49 = 0$ by factorization of perfect squares.

Solution

For a perfect square $4ac - b^2 = 0$

check if $x^2 - 14x + 49 = 0$ is a perfect square.

From the equation a = 1, b = –14, and c = 49.

$$4ac - b^2 = 4 \times 1 \times 49 - (-14)^2$$
$$= 196 - 196$$
$$= 0$$

Thus, $x^2 - 14x + 49 = 0$ is a perfect square.

That is $x^2 - 14x + 49 = \left(x - \frac{(-14)}{2}\right)^2 = 0$

$$= (x - 7)^2 = 0$$

Therefore, $x^2 - 14x + 49 = (x - 7)^2 = 0$ has identical roots $x = 7$.

Alternative method is as follows:

Comparing $x^2 - 14x + 49 = 0$ with $ax^2 + bx + c = 0$;

$a = 1, b = -14, c = 49$

For a perfect square;

$$x = \frac{-b}{2a}$$
$$= \frac{-(-14)}{2(1)}$$
$$= 7.$$

Exercise 3.4

Solve the following quadratic equations by the factorization method:

1. $4x^2 - 49 = 0$
2. $9y^2 - 6y + 1 = 0$
3. $y^2 - 4 = 0$
4. $a^2 - 8a + 16 = 0$
5. $25a^2 - 9 = 0$
6. $9x^2 + 12x + 4 = 0$
7. $16b^2 = 49$
8. $4d^2 - 20d + 25 = 0$
9. $x^2 = 1$
10. $x^2 = 10x - 25$
11. $(x-3)^2 - 25 = 0$
12. $\frac{x^2}{4} = 4$
13. $(x-5)^2 = 25$
14. $b^2 + \frac{2b}{5} + \frac{1}{25} = 0$
15. $c^2 + \frac{2}{7}c + \frac{1}{49} = 0$
16. $(x-9)^2 - 36 = 0$
17. $9x^2 = 12x - 4$
18. $\frac{1}{4}a^2 + a + 1 = 0$
19. $4r - 1 = 4r^2$
20. $x^2 - 5x + \frac{25}{4} = 0$

Solving quadratic equations by completing the square

It is useful to establish the relationship between the coefficients of different terms of a perfect square. Engage in the following activity to establish this relationship.

Activity 3.1: Deducing the relationship between coefficients a, b and c of a quadratic equation

Steps:

1. In pairs or groups, copy the following chart in your exercise book.

	ax^2+bx+c	a	b	c	$\frac{b}{2a}$	$\left(\frac{b}{2a}\right)^2$	$\left(x+\frac{b}{2a}\right)^2$
(a)	$x^2+5x+\frac{25}{4}$	1	5	$\frac{25}{4}$	$\frac{5}{2}$	$\frac{25}{4}$	$\left(x+\frac{5}{2}\right)^2$
(b)	x^2-6x+9						
(c)	$x^2-16x+64$						
(d)	$x^2-\frac{3}{2}x+\frac{9}{16}$						
(e)	x^2+4x+4						
(f)	x^2-2x+1						
(g)	$x^2-x+\frac{1}{4}$						
(h)	$x^2-\frac{3}{4}x+\frac{9}{84}$						
(i)	$x^2+10x+25$						
(j)	$x^2-20x+100$						

2. Fill in the empty spaces in the chart. Part (a) has been done as an example. Note that, the coefficient of x^2 is 1 in all cases.
3. Study the chart carefully then give the relationship between a, b and c.

If the chart is correctly filled, the relationship $c=\left(\frac{b}{2a}\right)^2$ can be established.

The relationship helps in completing the square when ax^2+bx is given.

In general, if $ax^2+bx+c=0$ is a perfect square then it should be possible to express it in the form $(x+k)^2=0$.

In order for $ax^2+bx+c=0$ to be equated to $(x+k)^2=0$ we first divide the equation $ax^2+bx+c=0$ throughout by a (a ≠ 0) to get $x^2+\frac{b}{a}x+\frac{c}{a}=0$.

We can now equate the two sides of the equation $x^2+\frac{b}{a}x+\frac{c}{a}=(x+k)^2$.

Expansion of the right hand side term gives $(x+k)^2=x^2+2kx+k^2$.

We equate this to $x^2+\frac{b}{a}x+\frac{c}{a}$ and get $x^2+\frac{b}{a}x+\frac{c}{a}=x^2+2kx+k^2$.

Comparing coefficients on both sides of this equation we get

$$\frac{b}{a}=2k,\ \frac{c}{a}=k^2.$$

Therefore, $k=\frac{b}{2a}$.

Hence, if $ax^2+bx+c=0$ is a perfect square, $\left(x+\frac{b}{2a}\right)^2=0$

and the two roots of the equation $ax^2+bx+c=0$ are $x=\frac{b}{2a}$ (repeated root).

Example 3.14

What must be added to x^2+10x to make the expression a perfect square?

Solution

The term to be added must be the square of half the coefficient of x.

The coefficient of x is 10.

Half of 10 is $\frac{1}{2}(10)=5$.

The square of 5 is 25.

Thus, 25 must be added to x^2+10x to make it a perfect square.

Therefore, x^2+10x into a perfect square is $x^2+10x+25$.

Example 3.15

What must be added to $x^2-\frac{5}{2}x$ to make it a perfect square?

Solution

The coefficient of x is $-\frac{5}{2}$.

Half of $-\frac{5}{2}$ is $\frac{1}{2}\left(-\frac{5}{2}\right)=-\frac{5}{4}$.

The square of $-\frac{5}{2}$ is $\left(-\frac{5}{4}\right)^2=\frac{25}{16}$.

Therefore, $\frac{25}{16}$ must be added to $x^2-\frac{5}{2}x$ to make it a perfect square;

that is $x^2-\frac{5}{2}x+\frac{25}{16}$.

This process of making an expression a perfect square is known as **completing the square**.

Exercise 3.5

To each of the following expressions add a term which will make it a perfect square and write the result in the form $(x+k)^2$.

1. x^2-12x
2. $a^2+\frac{3}{2}a$
3. $x^2+\frac{7}{2}x$
4. x^2-4x
5. x^2-x
6. x^2+5x
7. p^2+12p
8. $n^2+\frac{4}{3}n$
9. $t^2+\frac{t}{2}$
10. x^2+3x

The following are useful steps for solving quadratic equations of the form $ax^2+bx+c=0$ by completing the square:

1. Divide each term by a, that is, ensure the coefficient of x^2 is 1.
2. Move the term $\frac{c}{a}$ to the right side of the equation.
3. Complete the square on the left side of the equation and balance this by adding the same value to the right side of the equation.
4. Take the square root on both sides of the equation.
5. Subtract or add the number that remains on the left side of the equation in order to determine the values of x.

Example 3.16

Solve the equation $x^2+5x-14=0$ by completing the square.

Solution

The equation $x^2+5x-14=0$ has to be rearranged so that the left hand side (LHS) is converted to a perfect square.

$x^2+5x-14=0$,

$x^2+5x=14$ (adding 14 to both sides),

$$x^2+5x+\frac{25}{4}=14+\frac{25}{4},$$

(adding $\frac{25}{4}$ to both sides to make the LHS a perfect square).

Factorising the LHS to obtain,

$$\left(x+\frac{5}{2}\right)^2=\frac{56+25}{4},$$

$$\left(x+\frac{5}{2}\right)^2=\frac{81}{4},$$

$x+\frac{5}{2}=\pm\sqrt{\frac{81}{4}}$ (taking the square root of both sides),

$$x+\frac{5}{2}=\pm\frac{9}{2},$$

$$x=-\frac{5}{2}\pm\frac{9}{2},$$

$x=-\frac{5}{2}+\frac{9}{2}$ and $x=-\frac{5}{2}-\frac{9}{2}$,

$x=\frac{4}{2}=2$ and $x=-\frac{14}{2}=-7$.

Therefore, $x=2$ and $x=-7$.

Example 3.17

Solve the quadratic equation $x^2-5x+2=0$ by completing the square.

Solution

$x^2-5x+2=0,$

$x^2-5x=-2,$

$x^2-5x+\frac{25}{4}=-2+\frac{25}{4}$ (adding $x=\left(-\frac{5}{2}\right)^2$ or $\frac{25}{4}$ to both sides),

$\left(x-\frac{5}{2}\right)^2=\frac{-8+25}{4},$

$\left(x-\frac{5}{2}\right)^2=\frac{17}{4},$

$x-\frac{5}{2}=\pm\sqrt{\frac{17}{4}},$

$x=\frac{5}{2}\pm\sqrt{\frac{17}{4}},$

$x=\frac{5}{2}\pm\frac{\sqrt{17}}{2}.$

Therefore, $x=\frac{5}{2}+\frac{\sqrt{17}}{2}$ and $x=\frac{5}{2}-\frac{\sqrt{17}}{2}.$

Example 3.18

Solve the quadratic equation $3x^2-7x-6=0$ by completing the square.

Solution

$3x^2-7x-6=0$,

$x^2-\frac{7}{3}x-2=0$ (dividing each term by 3),

$x^2-\frac{7}{3}x=2$,

$x^2-\frac{7}{3}x+\frac{49}{36}=2+\frac{49}{36}$,

$\left(x-\frac{7}{6}\right)^2=\frac{72+49}{36}=\frac{121}{36}$,

$x-\frac{7}{6}=\pm\sqrt{\frac{121}{36}}=\pm\frac{11}{6}$,

$x=\frac{7}{6}\pm\frac{11}{6}$,

$x=\frac{7+11}{6}$ and $x=\frac{7-11}{6}$,

Therefore, $x=3$ and $x=-\frac{2}{3}$.

Example 3.19

Solve the quadratic equation $-3w^2 - 9w + 2 = 0$ by completing the square.

Solution

$-3w^2 - 9w + 2 = 0$.

$w^2 + 3w - \frac{2}{3} = 0$ (divide each term by -3)

$w^2 + 3w = \frac{2}{3}$.

$w^2 + 3w + \frac{9}{4} = \frac{2}{3} + \frac{9}{4}$.

$\left(w + \frac{3}{2}\right)^2 = \frac{35}{12}$.

$w + \frac{3}{2} = \pm\sqrt{\frac{35}{12}}$.

$w = -\frac{3}{2} \pm \sqrt{\frac{35}{12}}$.

$w = -\frac{3}{2} \pm \frac{\sqrt{35}}{2\sqrt{3}} = -\frac{3}{2} \pm \frac{\sqrt{105}}{6}$.

Therefore, $w = -\frac{3}{2} + \frac{\sqrt{105}}{6}$ and $-\frac{3}{2} - \frac{\sqrt{105}}{6}$.

Exercise 3.6

Solve the following quadratic equations by the method of completing the square.

1. $x^2+2x-15=0.$
2. $4v^2-8v+3=0.$
3. $3v^2+7v-6=0.$
4. $6-2c-c^2=0$
5. $x^2+2x-2=0.$
6. $12x^2-13x+3=0.$
7. $x^2-4x+2=0.$
8. $p^2+11=6p$
9. $3x^2+10x+6=0.$
10. $n^2+6n+2=0.$
11. $x^2+5x+3=0.$
12. $11-a-a^2=0$
13. $x^2+5x+2=0.$
14. $13-2g^2=-2g$
15. $m^2+11m+20=0.$
16. $9h^2=6h-1.$
17. $x^2-7x+11=0.$
18. $-3s^2-6s+1=0$
19. $x^2-11x-3=0.$
20. $14+15t-5t^2=0$

General formula for solving quadratic equation

The method of solving quadratic equations by completing the square can be used to derive the general formula for solving any quadratic equation.

The following are specific quadratic equations to illustrate the derivation of quadratic or general formula using the method of completing the square.

Example 3.20

Solve the quadratic equation $2x^2+7x+6=0$.

Solution

$2x^2+7x+6=0$,

Divide each term by 2,

$$x^2+\frac{7}{2}x+3=0.$$

Subtract 3 from both sides,

$$x^2+\frac{7}{2}x=-3.$$

Add $\left(\frac{7}{4}\right)^2$ on both sides,

$$x^2+\frac{7}{2}x+\left(\frac{7}{4}\right)^2=-3+\left(\frac{7}{4}\right)^2,$$

$$\left(x+\frac{7}{4}\right)^2=-3+\frac{49}{16}=\frac{-48+49}{16},$$

$$\left(x+\frac{7}{4}\right)^2=\frac{1}{16}.$$

Take the square root on both sides:

$$x+\frac{7}{4}=\pm\sqrt{\frac{1}{16}}=\pm\frac{1}{4},$$

Subtract $\frac{7}{4}$ on both sides

$$x=-\frac{7}{4}\pm\frac{1}{4}=\frac{-7\pm1}{4},$$

$$x=\frac{-7+1}{4}=\frac{-3}{2} \text{ and } x=\frac{-7-1}{4}=\frac{-8}{4}=-2.$$

Therefore, $x=-\frac{3}{2}$ and $x=-2$.

Example 3.21

Solve the quadratic equation $ax^2+bx+c=0$.

Solution

Divide each term by a

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Subtract $\frac{c}{a}$ from both sides,

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Add $\left(\frac{b}{2a}\right)^2$ on both sides:

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2.$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}.$$

Take the square root on both sides

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}},$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a},$$

Subtract $\frac{b}{2a}$ from both sides

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Therefore, $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$; where $a \neq 0$.

This gives the general formula for solving the quadratic equations.

That is, if $ax^2+bx+c=0$; then, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ provided that $b^2 \geq 4ac$.

The following are steps for solving quadratic equations of the form $ax^2+bx+c=0$ by the general formula:

1. Write the given quadratic equation in standard form $ax^2+bx+c=0$.
2. Compare the given quadratic equation with the standard form $ax^2+bx+c=0$ to identify the values of a, b and c.
3. Substitute the values of a, b and c in the formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and get the value of x.

Example 3.22

Solve $6x^2+11x+3=0$ using the general quadratic formula.

Solution

Comparing $ax^2+bx+c=0$ with $6x^2+11x+3=0$, gives the result: $a=6$, $b=11$ and $c=3$.

Substituting these values into $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, we get

$$x = \frac{-11\pm\sqrt{(11)^2-4\times6\times3}}{2\times6},$$

$$x = \frac{-11\pm\sqrt{121-72}}{12},$$

$$x = \frac{-11 \pm \sqrt{49}}{12}$$

$$= \frac{-11 \pm 7}{12}$$

$$x = -\frac{4}{12} \quad \text{and} \quad x = -\frac{18}{12}.$$

Therefore, $x = -\frac{1}{3}$ and $x = -\frac{3}{2}$.

Example 3.23

Solve $5x^2 - 6x - 1 = 0$ using the quadratic formula.

Solution

Since $5x^2 - 6x - 1 = 0$, then $a = 5$, $b = -6$ and $c = -1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 5 \times (-1)}}{2 \times 5},$$

$$x = \frac{6 \pm \sqrt{36 + 20}}{10} = \frac{6 \pm \sqrt{56}}{10},$$

$$x = \frac{6 + \sqrt{56}}{10} \quad \text{and} \quad x = \frac{6 - \sqrt{56}}{10},$$

$$x = \frac{6 + 2\sqrt{14}}{10} \quad \text{and} \quad x = \frac{6 - 2\sqrt{14}}{10}.$$

Therefore, $x = \frac{3 + \sqrt{14}}{5}$ and $x = \frac{3 - \sqrt{14}}{5}$.

Example 3.24

Solve the quadratic equation $-400k^2+317k-60=0$ by using the general quadratic formula.

Solution

$-400k^2+317k-60=0.$

Compare the given quadratic equation with the standard form $ax^2+bx+c=0$, hence, $a=-400$, $b=317$, and $c=-60$.

By using the general formula,

$$k=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Substitute the values of a, b and c,

$$k=\frac{-(317)\pm\sqrt{(317)^2-4(-400)(-60)}}{2(-400)},$$

$$k=\frac{-317\pm\sqrt{100\ 489-96\ 000}}{-800},$$

$$k=\frac{-317\pm\sqrt{4\ 489}}{-800}=\frac{-317\pm67}{-800}.$$

$$k=\frac{-317+67}{-800} \text{ and } k=\frac{-317-67}{-800}.$$

Therefore, $k=\frac{5}{16}$ and $k=\frac{12}{25}$.

Exercise 3.7

Solve the following quadratic equations using the general formula:

1. $x^2-4x+3=0$
2. $x^2+3x+1=0$
3. $3x^2-6x-2=0$
4. $2x^2+3x-2=0$
5. $2x^2-7x+3=0$
6. $(x+3)^2=10$
7. $3x^2-2x-2=0$
8. $-22n^2-15n+27=0$
9. $3x(x-1)=4$
10. $x(x+3)=(x-1)(2x-1)$
11. $2x^2-x-3=0$
12. $-145p^2+97p-6=0$
13. $x^2+2x-3=0$
14. $-12m^2+36m-27=0$
15. $400+20t-t^2=0$
16. $2e^2+e=6$
17. $6x^2+12x=0$
18. $\dfrac{x^2}{\left(1\frac{1}{2}-x\right)(2-x)}=4$

Word problems involving quadratic equations

The following are steps for solving word problems which involve quadratic equations:

1. Read the given problem carefully and choose a variable to represent the unknown quantity.
2. Formulate the quadratic equation from the given problem.
3. Solve the quadratic equation obtained.
4. Verify the solution.

Example 3.25

The difference between two positive numbers is 8 and their product is 105. Find the larger number.

Solution

Let the larger number be x.

Then, the smaller number is $x-8$.

Their product is $x(x-8)=105$.

$x^2-8x=105$,

$x^2-8x-105=0$,

$(x-15)(x+7)=0$

Hence, $x=15$ and $x=-7$.

But, since the required number is positive, -7 is rejected.

Hence, $x=15$

If the larger number is 15 then the smaller number is $15-y=8$, $y=7$.

Checking: Their product is $15 \times 7 = 105$.

Therefore, the larger number is 15.

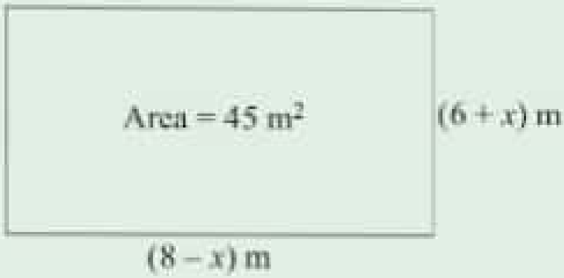
Example 3.26

A rectangular garden is 6 metres wide and 8 metres long. What length should be added to the shorter side and reduced from the longer side to form a rectangular garden with an area of 45 square metres?

Solution

Let x metres represent the added length. Then, the sides of the rectangular garden will be:

Width = $(6 + x)$ m and length = $(8 - x)$ as shown in the following figure.



Area of the new rectangular garden is given by: $(8 - x)(6 + x) = 45$,

$$(8-x)(6+x)=45$$
$$48+8x-6x-x^2=45$$
$$x^2-2x-3=0$$
$$x^2-3x+x-3=0$$
$$x(x-3)+1(x-3)=0$$
$$(x-3)(x+1)=0$$

$x-3=0$ and $x+1=0$.

$x=3$ and $x=-1$.

But length cannot be negative.

Hence, $x=-1$ is rejected.

Therefore, $x = 3$ metres

Verifying: New width $=(6+3)$ metres $=9$ metres

New length = $(8 - 3)$ metres = 5 metres

Therefore, area $=9$ metres $\times$ 5 metres $=45$ square metre.

Example 3.27

Juma bought a certain number of mangoes for 3 600 shillings. If each mango had costed 50 shillings less, he could have bought six more mangoes for the same amount of money, how many mangoes did he buy?

Solution

Let x represent the number of mangoes bought. Then the price of each mango was $\frac{3\,600}{x}$ shillings. Six more mangoes correspond to $(x + 6)$ mangoes.

Therefore, each mango would cost $\frac{3\,600}{x+6}$.

This price per mango is less than the previous one by 50 shillings.

Then, $\frac{3\,600}{x} - \frac{3\,600}{x+6} = 50$.

$3\,600(x+6) - 3\,600x = 50x(x+6)$.

$3\,600x + 21\,600 - 3\,600x = 50x^2 + 300x$

$21\,600 = 50x^2 + 300x$

Divide each term by 50 to obtain the following:

$x^2 + 6x - 432 = 0$,

$(x - 18)(x + 24) = 0$

$x - 18 = 0$ and $x + 24 = 0$.

$x = 18$ and $x = -24$.

Therefore, the number of mangoes bought was 18 because it is impossible to have a negative number of mangoes.

Checking: With 18 mangoes each costs 200 shillings. Six more mangoes each costs 150 shillings.

The difference is (200 – 150) shillings = 50 shillings.

Exercise 3.8

Answer the following questions:

1. The base of a triangle is 5 cm less than the height and the area is 33 cm^2. Find the length of the base.
2. The length of a classroom floor is 4 metres more than the width and the area is 221 square metres. Find the dimensions of the floor.
3. Find two positive numbers which differ by 5 and whose product is 126.
4. The perimeter of a rectangular garden is 60 metres and its area is 209 square metres. Find the dimensions of the garden.
5. Find two consecutive numbers whose product is 132.
6. Find two consecutive even numbers whose product is 80.
7. The ages of a man and his son are 35 and 9 years, respectively. How many years ago was the product of their ages 87 years?
8. A square garden of side 20 metres is surrounded by a path whose area is the same as that of the garden. Find the width of the path.
9. A piece of wire 40 cm long is cut into two parts and each part is then bent into a square. If the sum of the areas of these squares is 68 square centimetres, find the lengths of the two pieces of wire.

Chapter summary

1. To solve a quadratic equation means to find its solutions.
2. The solutions of a quadratic equation are also known as roots. To find the roots:
   (a) Arrange the equation in the form $ax^2 + bx + c = 0$
   (b) Factorize the left hand side if possible.
3. If the quadratic equation $ax^2 + bx + c = 0$ cannot be factorized, solve the equation either by completing the square or by applying the quadratic formula.

Revision exercise 3

Answer the following questions:

1. Solve the following quadratic equations by the factorization method:

(a) $x^2+3x=0$ (b) $3x^2-15x=0$

(c) $x=3x^2$ (d) $2x^2=3x$

(e) $x(5-x)=0$ (f) $7x^2-3x=0$

(g) $x^2+3x-40=0$ (h) $3x^2-7x-6=0$

(i) $12x^2+13x+3=0$ (j) $x^2+3x+2=0$

(k) $x^2-10x+24=0$ (l) $2x^2-x-6=0$

(m) $3x+2=9x^2$ (n) $-3x^2+11x-10=0$

(o) $4x^2=25$ (p) $y^2-36=0$

(q) $(x-8)^2=36$ (r) $4x^2=20x-25$

(s) $9y^2-6y+1=0$ (t) $(x+3)^2-49=0$

2. Solve the following quadratic equations by completing the square:

(a) $x^2+6x+7=0$ (b) $x^2-11x+1=0$

(c) $x^2=7x-7$ (d) $2x^2-10x+7=0$

(e) $m^2+5m=1$ (f) $p^2-10p+5=0$

(g) $2b^2=8b+11$ (h) $3a^2-12a=2$

(i) $5n^2=20n+28$ (j) $c^2-8c+13=0$

3. Use the quadratic formula to solve the following equations:

(a) $4x^2 - 4x + 1 = 0$ (b) $5x^2 + 12x + 3 = 0$

(c) $(3x-2)(2x-5) = 5x(x-2)$ (d) $3x^2 = 4x + 4$

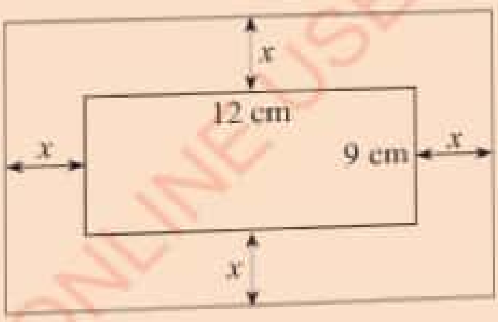
(e) $2x^2 - 5x + 2 = 0$ (f) $5x^2 - x - 18 = 0$

4. A man is 4 times as old as his son. In 4 years to come the product of their ages will be 520. Find the son's present age.

5. Sada has 1 800 shillings to buy pencils. There are two types of pencils whose prices differ by 40 shillings. If she buys the cheaper type she will get 12 more pencils than if she buys the expensive type. What are the prices of the two types of pencils?

6. Find two consecutive numbers such that the sum of their squares is equal to 145.

7. A picture measures 12 cm by 9 cm and is surrounded by a frame whose area is 180 square centimetres. Find the value of x as shown in the following figure.



8. When 6 is divided by a certain number, the result is the same as when 5 is added to the number and that sum divided by 6. Find the number.

9. Find the whole number such that twice its square is 11 more than 21 times the number.

10. A piece of wire 56 cm long is bent to form a rectangle of area 171 cm^2. Find the dimensions of the rectangle.

Chapter Four

Logarithms

Introduction

There are several ways of simplifying mathematical expressions. A simple way of adding one number many times repeatedly is to use multiplication. Similarly, a simple way of multiplying one number many times repeatedly is to use exponents. Likewise, the simple way of dealing with exponents is to use logarithms. The word logarithm originates from two Greek words; "logs" means ratio and "arithmos" which means number. In this chapter you will learn to write numbers in standard form (scientific notation), use laws of logarithms in calculations, and to find values of logarithms by using a table of common logarithms. The competences developed in this chapter will enable you to simplify a process of solving different scientific and engineering problems, including evaluation of complicated mathematical expressions involving numbers.

Standard form

When a number is expressed in the form $A \times 10^n$, where $1 \leq A < 10$ and n is an integer, it is said to be in a **Standard form or Scientific notation.** For example, the following numbers are expressed in the standard form:

(a) $290 = 2.9 \times 100 = 2.9 \times 10^2$

(b) $29 = 2.9 \times 10 = 2.9 \times 10^1$

(c) $2.9 = 2.9 \times 10^0$

(d) $0.29 = 2.9 \times 0.1 = 2.9 \times \frac{1}{10} = 2.9 \times 10^{-1}$

When writing numbers in the standard form, the following should be noted:

1. For numbers between 0 and 1, the decimal point is moved to the right and the exponent of 10 is negative.
2. For numbers greater than or equal to 10, the decimal point is moved to the left and the exponent of 10 is positive.
3. The sign of the exponent of 10 is determined by the number of places where the decimal point moves to the right or to the left.
4. For numbers which are greater than or equal to 1 and less than 10, the exponent of 10 is 0.
5. In multiplication of exponential numbers, the exponents are added, while in division the exponents are subtracted.

Example 4.1

Write the following numbers in the standard form:

(a) 230 000 000 (b) 245 (c) 0.00045

Solution

(a) $230\ 000\ 000 = 2.3 \times 100\ 000\ 000 = 2.3 \times 10^8$.

(b) $245 = 2.45 \times 100 = 2.45 \times 10^2$.

(c) $0.00045 = 4.5 \times 0.0001 = 4.5 \times 10^{-4}$.

Example 4.2

Write each of the following standard forms in decimal numerals:

(a) 4.8×10^3 (b) 3.5×10^{-6} (c) 1.4×10^{-4}

Solution

(a) $4.8 \times 10^3 = 4.8 \times 1000 = 4800$

(b) $3.5 \times 10^{-6} = 3.5 \times 0.000001 = 0.0000035$

(c) $1.4 \times 10^{-4} = 1.4 \times 0.0001 = 0.00014$

Example 4.3

Simplify each of the following and give the answers in the standard form:

(a) $(7 \times 10^2)(8 \times 10^4)$ (b) $(7 \times 10^{-2})(8 \times 10^{-4})$

Solution

(a) $(7\times10^2)(8\times10^4) = (7\times8)\times(10^2\times10^4)$

$= 56\times10^6$

$= 5.6\times10\times10^6$

$= 5.6\times10^7$

Therefore, $(7 \times 10^2)(8 \times 10^4) = 5.6 \times 10^7$.

(b) $(7\times10^{-2})(8\times10^{-4}) = 7\times8\times10^{-2}\times10^{-4} = 56\times10^{-6}$

$= 56\times10^{-6}$

$= 5.6\times10^1\times10^{-6}$

$= 5.6\times10^{-5}$

Therefore, $(7 \times 10^{-2})(8 \times 10^{-4}) = 5.6 \times 10^{-5}$.

Example 4.4

Calculate the following numbers giving the answer in the standard form:

(a) $\frac{4\times10^{8}}{5\times10^{5}}$ (b) $\frac{4\ 848\times10^{-5}}{20\times10^{2}}$

Solution

$$\frac{4\times10^{8}}{5\times10^{5}}=\frac{4}{5}\times\frac{10^{8}}{10^{5}}$$
$$=0.8\times10^{3}$$
$$=8.0\times10^{-1}\times10^{3}$$
$$=8.0\times10^{2}$$

Therefore, $\frac{4\times10^{8}}{5\times10^{5}}=8.0\times10^{2}$.

Solution

$$\frac{4\ 848\times10^{-5}}{20\times10^{2}}=\frac{4\ 848}{20}\times\frac{10^{-5}}{10^{2}}$$
$$=242.4\times10^{-7}$$
$$=242.4\times10^{-7}$$
$$=2.424\times10^{2}\times10^{-7}$$
$$=2.424\times10^{-5}$$

Therefore, $\frac{4\ 848\times10^{-5}}{20\times10^{2}}=2.424\times10^{-5}$.

Exercise 4.1

Answer the following questions:

1. Write each of the following numbers in the standard form:

(a) 31 065 (b) 95.1 (c) 9 999 (d) 6
(e) $\frac{1}{100}$ (f) 69.03 (g) 253 009 115 (h) 5.41
(i) 0.0004068 (j) 7.245 (k) 1 985 (l) 0.000008
(m) $\frac{3}{4}$ (n) 30 (o) 463.18 (p) 26.5×10^{4}

2. Write in decimal numerals each of the following:

(a) 9.10×10^{5} (b) 7.4×10^{-4} (c) 3×10^{0}
(d) 26.5×10^{4} (e) 2.74×10^{4} (f) 4.2×10^{4}
(g) 3.68×10^{3} (h) 8.67×10^{-2} (i) 2.5×10^{1}
(j) 9.18×10^{5} (k) 4.0×10^{1} (l) 1.06×10^{2}

3. Evaluate each of the following expressions and write your answer in the standard form:

   (a) $\left(2.25\times10^{3}\right)\times\left(4\times10^{6}\right)$ (b) $\left(2.75\times10^{4}\right)\times\left(8\times10^{2}\right)$

   (c) $\left(8.5\times10^{-3}\right)\times\left(2.4\times10^{2}\right)$ (d) $\left(25\times10^{4}\right)\times\left(8\times10^{4}\right)$

   (e) $\left(222\times10^{-3}\right)\times\left(5.5\times10^{-2}\right)$

4. Evaluate each of the following expressions, giving your answer in the form $A\times10^{n}$, (where $1\leq A<10$, and n is an integer):

   (a) $\frac{9\times10^{4}}{2\times10^{2}}$ (b) $\frac{7\times10^{6}}{1.4\times10^{3}}$ (c) $\frac{3\times10^{5}}{5\times10^{8}}$ (d) $\frac{3\times10^{8}}{5\times10^{5}}$

   (e) $\frac{192\times10^{-3}}{3\times10^{-2}}$ (f) $\frac{1984\times10^{-8}}{400\times10^{-3}}$ (g) $\frac{3.5\times10^{-3}}{7\times10^{3}}$ (h) $\frac{125\times10^{-2}}{5\times10^{2}}$

5. Find the area of a circle of radius 20 cm giving the answer in the standard form (use $\pi=3.142$).

6. The distance of the Earth from the sun is about 1.494×10^{11} kilometres. Write this distance as a decimal numeral.

7. The diameters of certain molecules have been calculated in centimetres as follows:

   (a) Hydrogen: 2.47×10^{-8}

   (b) Oxygen: 3.39×10^{-8}

   Write these diameters in decimal form.

Math F2 - (BOOK) (DUMMY) Sept 4.indb 113

Logarithms

When a number is expressed in power form, it is written as a base raised to an exponent. For example, if $a = b^x$ then 'a' is written in terms of base 'b' raised to an exponent 'x' or 'x' is the logarithm of a to base b.

The exponent x is the number that shows how many times a base is multiplied by itself to produce a product. Thus x is called the **logarithm** of a to base b.

In short form, it is written as $\log_b a = x$ where $a > 0$, and this is called the **logarithmic notation**.

For instance, 64 in exponential form can be expressed as 2^6. The exponent 6 is called the logarithm of 64 to base 2. In short it is written as $6 = \log_2 64$.

For example:

$25 = 5^2$ is written as $2 = \log_5 25$

$1000 = 10^3$ is written as $3 = \log_{10} 1000$

$0.0001 = 10^{-4}$ is written as $-4 = \log_{10} 0.0001$

In general, $a = b^x$ is written as $x = \log_b a$ where $a > 0$, x is a real number and $b > 0$.

Note that, when stating the logarithm of a number, the base to which it is raised should be stated.

Consider the following:

$64 = 2^6$ means 6 is the logarithm of 64 to base 2 written as $\log_2 64 = 6$

$64 = 4^3$ means 3 is the logarithm of 64 to base 4 written as $\log_4 64 = 3$

$64 = 8^2$ means 2 is the logarithm of 64 to base 8 written as $\log_8 64 = 2$

Example 4.5

Express each of the following expressions according to the given instruction:

(a) $\log_2 8 = 3$ in exponential form

(b) $5^{-3} = \frac{1}{125}$ in logarithmic form

(c) $0.1 = 10^{-1}$ in logarithmic form

Solution

(a) $\log_2 8 = 3$ in exponential form is $2^3 = 8$

(b) $5^{-3} = \frac{1}{125}$ in logarithmic form is $-3 = \log_5\left(\frac{1}{125}\right)$

(c) $0.1 = 10^{-1}$ in logarithmic form is $-1 = \log_{10}(0.1)$

Example 4.6

Solve for x each of the following expressions:

(a) $x = \log_{10} 100$ (b) $-5 = \log_x(0.00001)$ (c) $\log_{10} x = 2$

Solution

(a) $x = \log_{10} 100$ writing in exponential form, which gives $100 = 10^x$ and $10^2 = 10^x$

Therefore, $x = 2$.

(b) $-5 = \log_x 0.00001$ writing in exponential form, which gives $x^{-5} = 0.00001$

But $0.00001 = 10^{-5}$

So that $x^{-5} = 10^{-5}$

Therefore, $x = 10$.

(c) $\log_8 x = 2$, writing in exponential form, which gives $8^2 = x$

Therefore, $x = 64$.

Special cases on logarithms

The following are some special cases on logarithms of numbers:

1. If $\log_a a = x$, then, $a^x = a^1$ which gives $x = 1$.

   Therefore, $\log_a a = 1$.

   Thus, $\log_{10} 10 = 1$ and $\log_2 2 = 1$.

2. If $\log_a (a)^n = x$ for a positive number a; then $a^x = a^n$, which gives $x = n$

   Therefore, $\log_a (a)^n = n$.

Base 10 logarithms

Base 10 logarithms are logarithms of numbers to base 10, sometimes known as **common logarithms**. The base 10 is usually left out when writing logarithms to base 10. For example, instead of writing $\log_{10} 315$, it is simply written as log 315. In general, instead of writing $\log_{10} x$ it is written as $\log x$.

The logarithms of numbers which are powers of integral exponents of 10 can be found as follows:

$\log 100 = \log 10^2 = 2$

$\log 10 = \log 10^1 = 1$

$\log 1 = \log 10^0 = 0$

$\log 0.1 = \log 10^{-1} = -1$

$\log 0.01 = \log 10^{-2} = -2$

$\log 0.001 = \log 10^{-3} = -3$

In general, $\log 10^n = n$

Exercise 4.2

Answer the following questions:

1. Complete the following:

    (a) $2^4 = 16$, so $4 = \log ...$

    (b) $5^2 = 25$, so $2 = \log ...$

    (c) $3^5 = 243$, so $5 = \log ...$

2. Write the following exponential forms in logarithmic forms:

    (a) $4^3 = 64$ (b) $10^6 = 1\,000\,000$ (c) $3^{-2} = \frac{1}{9}$

    (d) $10^0 = 1$ (e) $13^1 = 13$ (f) $10^{-3} = 0.001$

    (g) $10 = \frac{1}{10^{-1}}$ (h) $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$ (i) $23^{-1} = \frac{1}{23}$

3. Write the following logarithmic forms in exponential forms:

    (a) $\log_{11} 121 = 2$ (b) $\log_{10} 10\,000 = 4$ (c) $\log_{10} 0.1 = -1$

    (d) $\log_4 2 = \frac{1}{2}$ (e) $\log_2 0.25 = -2$ (f) $\log_{\frac{1}{5}}\left(\frac{1}{125}\right) = -3$

4. Find the value of x in each of the following equations:

    (a) $\log_2 x = 2$ (b) $\log_5 1 = x$ (c) $\log_5 x = 1$

    (d) $\log_4 x = -3$ (e) $\log_4 256 = x$ (f) $\log_x 10 = 1$

    (g) $\log_2\left(\frac{1}{1\,024}\right) = x$ (h) $\log_x 100\,000\,000 = 8$ (i) $\log_x 1\,000 = 3$

    (j) $\log_{\frac{1}{2}} 0.0625 = x$ (k) $\log_{25} x = \frac{3}{2}$ (l) $\log_x \frac{1}{27} = -3$

    (m) $\log_2 256 = x$ (n) $\log_2 1 = x$ (o) $\log_2 x = \frac{7}{2}$

5. Determine the number whose logarithm to base 5 is -3.

Laws of logarithms

There are rules which guide us to write expressions involving logarithms in different ways. These rules are known as the **Laws of logarithms**. The laws apply to logarithms to any base. However, the same base must be used throughout for all steps. The following are the four rules of logarithms:

1. Product rule	$\log_a(xy) = \log_a x + \log_a y$
2. Quotient rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. Power rule or Rule of exponents	$\log_a x^n = n\log_a x$
4. Roots rule	$\log_a \sqrt[m]{x^n} = \log_a x^{\frac{n}{m}} = \frac{n}{m}\log_a x$ where $m \neq 0$, n and m are integers.

Derivation of the laws of logarithms

Product law

Let $p = \log_a x$ and $q = \log_a y$ (1)

Expressing equation (i) in exponential form gives

$x = a^p$ and $y = a^q$ (2)

From the rules of exponents $a^p \times a^q = a^{p+q}$

It follows that $xy = a^p \times a^q = a^{p+q}$ (3)

Expressing equation (iii) in logarithmic form gives

$$\log_a(xy) = \log_a a^{p+q}$$
$$= (p+q)\log_a a$$
$$= p+q$$

$\log_a(xy) = p+q$ (4)

Substituting the expressions for p and q from equation (1) in (4) to obtain

$\log_a(xy) = \log_a x + \log_a y$.

Example 4.7

Evaluate

(a) $\log_3(81\times9)$ (b) $\log_5(125\times625)$

Solution

(a)
$$\begin{aligned}\log_3(81\times9) &= \log_3 81+\log_3 9\\ &= \log_3 3^4+\log_3 3^2\\ &= 4\log_3 3+2\log_3 3\\ &= (4\times1)+(2\times1)\\ &= 4+2\\ &= 6\end{aligned}$$

Therefore, $\log_3(81\times9)=6$.

(b)
$$\begin{aligned}\log_5(125\times625) &= \log_5 125+\log_5 625\\ &= \log_5 5^3+\log_5 5^4\\ &= 3\log_5 5+4\log_5 5\\ &= (3\times1)+(4\times1)\\ &= 3+4\\ &= 7\end{aligned}$$

Therefore, $\log_5(125\times625)=7$.

Example 4.8

Find: (a) $\log_2(4\times8)$ (b) $\log_{10}(0.01\times100\ 000)$

Solution

(a)
$$\begin{aligned}\log_2(4\times8) &= \log_2 4+\log_2 8\\ &= \log_2 2^2+\log_2 2^3\\ &= 2\log_2 2+3\log_2 2\\ &= (2\times1)+(3\times1)\\ &= 2+3\\ &= 5\end{aligned}$$

Therefore, $\log_2(4\times8)=5$.

(b)
$$\begin{aligned}&\log_{10}(0.01\times100\ 000)\\ &= \log_{10} 0.01+\log_{10} 100\ 000\\ &= \log 10^{-2}+\log 10^5\\ &= -2\log 10+5\log 10\\ &= (-2\times1)+(5\times1)\\ &= -2+5\\ &= 3\end{aligned}$$

Therefore, $\log_{10}(0.01\times100000)=3$.

Math F2 – (BOOK) (DUMMY) Sept 4.indb 119

Power law

Let, $p = \log_a m^n$

$= \log_a (m \times m \times m \times \cdots \times m)$ (up to n times)

$= \log_a m + \log_a m + \log_a m + \cdots + \log_a m$ (by using logarithm product law)

$= n\log_a m$

Therefore, $\log_a m^n = n\log_a m$

Example 4.9

Find $\log_3 9^2$

Solution

$$\begin{aligned}\log_3 9^2 &= 2\log_3 9\\ &= 2\log_3 3^2\\ &= 2(2)\log_3 3\\ &= 4(1)\\ &= 4\end{aligned}$$

Therefore, $\log_3 9^2 = 4$.

Example 4.10

Find the value of:

(a) $\log_4 (64)^5$ (b) $\log\ (100)^{25}$ (c) $\log\ (0.1)^6$

Solution

(a) $$\begin{aligned}\log_4 (64)^5 &= 5\log_4 (64)\\ &= 5\log_4 4^3\\ &= 5\times 3\log_4 4\\ &= 15\end{aligned}$$

Therefore, $\log_4 (64)^5 = 15$.

(b) $$\begin{aligned}\log(100)^{25} &= 25\log 100\\ &= 25\log 10^2\\ &= 25\times 2\log 10\\ &= 25\times 2\times 1\\ &= 50\end{aligned}$$

Therefore, $\log(100)^{25} = 50$.

(c) $\log(0.1)^6 = 6\log(0.1)$

$= 6\log 10^{-1}$

$= 6\times(-1)\log 10$

$= 6\times(-1)\times 1$

$= -6$

Therefore, $\log(0.1)^6 = -6$.

Quotient rule

Let, $p = \log_a x$ and $q = \log_a y$ (1)

Expressing equation (1) in exponential form, gives

$x = a^p$ and $y = a^q$ (2)

From the rules of exponents $a^p \div a^q = a^{p-q}$.

From equation (2) divide x by y, to obtain

$\frac{x}{y} = a^p \div a^q = a^{p-q}$ (by laws of exponents)

$\frac{x}{y} = a^{p-q}$ (3)

Express equation (3) in logarithmic form to obtain

$\log_a\left(\frac{x}{y}\right) = \log_a a^{p-q}$

$= (p-q)\log_a a$

$= p-q$ (4)

Substitute the expressions for p and q from equation (1) in (4) to obtain

$\log_a x - \log_a y$

Therefore, $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

Example 4.11

Find the value of $\log_3\left(\frac{27}{9}\right)$.

Solution

$$\log_3\left(\frac{27}{9}\right) = \log_3 27 - \log_3 9$$
$$= \log_3 3^3 - \log_3 3^2$$
$$= 3\log_3 3 - 2\log_3 3$$
$$= 3(1) - 2(1)$$
$$= 1$$

Therefore, $\log_3\left(\frac{27}{9}\right) = 1$.

Example 4.12

Evaluate the following:

(a) $\log_3(9 \div 243)$ (b) $\log(10 \div 0.001)$

Solution

(a) $$\log_3(9 \div 243) = \log_3 9 - \log_3 243$$
$$= \log_3 3^2 - \log_3 3^5$$
$$= 2\log_3 3 - 5\log_3 3$$
$$= 2(1) - 5(1)$$
$$= -3$$

Therefore, $\log_3(9 \div 243) = -3$.

(b) $$\log(10 \div 0.001) = \log_{10}\left(\frac{10}{0.001}\right)$$
$$= \log_{10} 10 - \log_{10} 0.001$$
$$= \log_{10} 10^1 - \log_{10} 10^{-3}$$
$$= 1\log_{10} 10 - (-3)\log_{10} 10$$
$$= 1 + 3 = 4$$

Therefore, $\log(10 \div 0.001) = 4$.

Roots law

Let, $p = \log_a \sqrt[n]{x}$.

From the law of exponents, $\sqrt[n]{x} = x^{\frac{1}{n}}$

It follows that,

$$p = \log_a x^{\frac{1}{n}}$$
$$= \frac{1}{n}\log_a x$$

Therefore, $\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$.

Example 4.13

Find the value of each of the following:

(a) $\log_9 \sqrt{729}$

(b) $\log \sqrt[3]{0.000001}$

(c) $\log \sqrt[5]{100000}$

(d) $\log_3 \sqrt{\frac{1}{27}}$

Solution

(a)
$$\log_9 \sqrt{729} = \log_9 (729)^{\frac{1}{2}}$$
$$= \frac{1}{2}\log_9 9^3$$
$$= \frac{1}{2}\times 3\log_9 9$$
$$= \frac{3}{2}$$

Therefore, $\log_9 \sqrt{729} = \frac{3}{2}$.

(b)
$$\log \sqrt[3]{0.000001} = \log(0.000001)^{\frac{1}{3}}$$
$$= \log(10^{-6})^{\frac{1}{3}}$$
$$= \log 10^{-\frac{6}{3}}$$
$$= \log 10^{-2}$$
$$= -2\log 10$$
$$= -2\times 1$$

Therefore, $\log \sqrt[3]{0.000001} = -2$.

Math F2 - (BOOK) (DUMMY) Sept 4.indb 123

(c) $\log \sqrt[5]{100\,000} = \log(100\,000)^{\frac{1}{5}}$

$= \frac{1}{5}\log 100\,000$

$= \frac{1}{5}\log 10^5$

$= \frac{1}{5} \times 5 \log 10$

$= \frac{1}{5} \times 5 \times 1$

$= 1.$

(d) $\log_3 \sqrt{\frac{1}{27}} = \log_3 \left(\frac{1}{27}\right)^{\frac{1}{2}}$

$= \frac{1}{2}\log_3 \left(\frac{1}{27}\right)$

$= \frac{1}{2}\log_3 \left(\frac{1}{3}\right)^3$

$= \frac{1}{2} \times 3\log_3 \left(\frac{1}{3}\right)$

$= \frac{1}{2} \times 3\log_3 3^{-1}$

$= \frac{1}{2} \times 3 \times (-1)\log_3 3$

$= \frac{1}{2} \times 3 \times (-1) \times 1$

$= -\frac{3}{2}$

$\therefore \log_3 \sqrt{\frac{1}{27}} = -\frac{3}{2}.$

Example 4.14

Determine the value of x if $\log_2 x = \log_2 2 - \log_2 3$.

Solution

$\log_2 x = \log_2 2 - \log_2 3$;

$\log_2 x = \log_2 \left(\frac{2}{3}\right)$

$x = \frac{2}{3}$

Therefore, $x = \frac{2}{3}$.

Example 4.15

Given that $\log 2 = 0.30103$ and $\log 3 = 0.47712$ calculate the value of log 48 and write the laws involved at each stage of finding the answer.

Solution

$\log 48 = \log(2^4 \times 3)$ (factors of 48)

$= \log 2^4 + \log 3$ (logarithm of a product)

$= 4\log 2 + \log 3$ (logarithm of a power)

$= 4(0.30103) + 0.47712$

$= 1.68124$

Therefore, $\log 48 = 1.68124$.

Example 4.16

Find the value of x if $\log_a x = \frac{1}{2}\log_a 4 + \frac{1}{3}\log_a 27$.

Solution

$\log_a x = \frac{1}{2}\log_a 4 + \frac{1}{3}\log_a 27$

$= \log_a 4^{\frac{1}{2}} + \log_a 27^{\frac{1}{3}}$

$= \log_a (2^2)^{\frac{1}{2}} + \log_a (3^3)^{\frac{1}{3}}$

$= \log_a 2 + \log_a 3$

$= \log_a (2 \times 3)$

$= \log_a 6$

$\log_a x = \log_a 6$

$\therefore x = 6$

Example 4.17

Simplify $\frac{\log 6}{\log 216}$.

Solution

$$\frac{\log 6}{\log 216} = \frac{\log 6}{\log 6^3}$$

$$= \frac{\log 6}{3\log 6}$$

$$= \frac{1}{3}$$

Therefore, $\frac{\log 6}{\log 216} = \frac{1}{3}$.

Exercise 4.3

Answer the following questions:

1. Find the value of each of the following:

   (a) $\log_3 (9 \times 81)$ (b) $\log_5 (5 \times 25 \times 625)$

   (c) $\log\ (100 \div 0.0001)$ (d) $\log_7 (49 \div 343)$

2. Calculate each of the following:

   (a) $\log_7 49^3$ (b) $\log_5 (5 \div 125)^3$

   (c) $\log \sqrt[3]{0.0001}$ (d) $\log_2 \sqrt{8}$

   (e) $\log \sqrt[3]{1000}$ (f) $\log 0.001^4$

3. Simplify each of the following using the laws of logarithms:

   (a) $\frac{\log 64}{\log 4}$ (b) $\log_2 28 - \log_2 7$

   (c) $\log_3 10 + \log_3 8.1$ (d) $\log 20 + \log 50$

   (e) $\log 3^4 + \log\left(\frac{10}{81}\right)$ (f) $\frac{\log \sqrt[3]{4}}{\log \sqrt{4}}$

4. Find without using tables or a calculator, the value of $2\log 5 + \log 36 - \log 9$.

5. Determine the value of x in each of the following equations:

   (a) $\log_3 x = \log_3 4 + \log_3 5$ (b) $\log x = \log 2 + \log 5$

6. Find the value of x in each of the following equations:

   (a) $\log x = \log 20 - \log 200 + \log 50$ (b) $\log x = \log 2 - \log 20 + \log 5$

7. Find the value of x in each of the following equations:

   (a) $\log_a x = \log_a 5 + 2\log_a 3$ (b) $\log_a x = 3\log_a 15 - 2\log_a 15$

8. Given that $\log 2 = 0.30103$, $\log 3 = 0.47712$ and $\log 5 = 0.69897$.

   Find the value of $\log 90$.

9. If $\log y + 2\log x = 3$, express y in terms of x.

Logarithms of numbers which are between 1 and 10

By expressing a number in standard form and then using the laws of logarithms, the logarithm of any number between 1 and 10 can be found.

For example, the logarithm of 347 is calculated as follows;

$$\begin{aligned}\log 347 &= \log\left(3.47\times 10^2\right)\\ &= \log 3.47 + \log 10^2\\ &= \log 3.47 + 2\end{aligned}$$

Since log 1 = 0 and log 10 = 1, then the logarithm of 3.47 is between 0 and 1. That is, $0 < \log 3.47 < 1$.

The value of log 3.47 can be read from the table of common logarithms which gives the approximate values of logarithms of numbers between 1 and 10 only. That is why it is necessary to express a number in standard form.

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Tables of common logarithms

The four figure logarithmic tables are widely used to find values of logarithms. These tables gives the approximate values of logarithms from the equation $y = \log_{10} x$, for $1 < x < 10$.

In order to find the logarithm of 3.47 from the table of common logarithms, first locate 3.4 on the extreme left hand column with x values (see Table 4.1). The row with 3.4 meets the column labeled 7 at 0.5403.

$\therefore \log 3.47 = 0.5403$ (since log 3.47 lies between 0 and 1)

Therefore, $\log 347 = \log 3.47 + 2 = 0.5403 + 2 = 2.5403$.

Table 4.1: Common logarithms

$\log_{10} x$ or $\log x$																	
								Mean Differences (Add)									
x	0	1	...	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	...	0.0253	0.0294	0.0334	0.0374	4	8	12	17	21	25	29	33	37
⋮																	
3.3	0.5185	0.5198		0.5263	0.5276	0.5289	0.5302	1	3	4	5	6	8	9	10	12
3.4	0.5315	0.5328		0.5391	0.5403	0.5416	0.5428	1	3	4	5	6	8	9	10	11
3.5	0.5441	0.5453		0.5514	0.5527	0.5539	0.5551	1	2	4	5	6	7	9	10	11

The logarithm of a number is formed by two parts, an integral part and a decimal part. The integral part is called the **characteristic** and the decimal part is called the **mantissa.** For example, $\log 347 = 2.5403$, the characteristic is 2 and the mantissa is 5403.

The logarithm of 34.75 can be found from logarithm tables as follows: First, read and record the logarithm of 3.47; then continue moving horizontally along the row that contains 3.4 on the extreme left until you reach the column with 5 at the top in the differences column. The number 6 obtained is added to the last digit of 5403 (the mantissa) to obtain 5409.

Therefore, $\log 34.75 = 1.5409$. Here the characteristic is 1 and mantissa is 5409.

To find the logarithm of a number with more than four figures, first write the number in standard form, then round up the number to four significant figures and then follow the procedures of finding the logarithm of a number by using common logarithm tables.

Example 4.18

Find the logarithm of each of the following numbers:

(a) 356 (b) 2 534 (c) 62.94 (d) 75 648 (e) 64.667

Solution

(a) $\log 356 = \log\left(3.56 \times 10^2\right)$

$= \log 3.56 + \log 10^2$ (logarithm of a product)

$= 0.5514 + 2$ (logarithm of a power)

$= 2.5514$

Therefore, $\log 356 = 2.5514$.

(b) $\log 2534 = \log\left(2.534 \times 10^3\right)$

$= \log 2.534 + \log 10^3$ (logarithm of a product)

$= 0.4038 + 3$ (logarithm of a power)

$= 3.4038$

Therefore, $\log 2534 = 3.4038$.

(c) $\log 62.94 = \log\left(6.294 \times 10^1\right)$

$= \log 6.294 + \log 10^1$ (logarithm of a product)

$= 0.7990 + 1$ (logarithm of a power)

$= 1.7990$

Therefore, $\log 62.94 = 1.7990$.

(d) $\log 75\,648 = (7.5648 \times 10^4)$

$= \log 7.5648 + \log 10^4$ (logarithm of a product)

$= 0.8788 + 4$ (logarithm of a power)

$= 4.8788$

Therefore, $\log 75\,648 = 4.8788$.

(e) $\log 64.667 = \log (6.4667 \times 10^1)$

$= \log 6.4667 + \log 10^1$ (logarithm of a product)

$= 0.8107 + 1$ (logarithm of a power)

$= 1.8107$

Therefore, $\log 64.667 = 1.8107$.

Antilogarithms

Since 1.7990 is the logarithm of 62.94, then 62.94 is called the antilogarithm of 1.7990. This can also be said that the antilog of 1.7990 is 62.94. Tables of antilogarithms can be used to find a number whose logarithm is known. For example, to obtain the number (antilogarithm) whose logarithm is 1.7990, the following steps should be followed:

Step 1: In the extreme left column of Table 4.2 locate the number .79.

Step 2: Move horizontally along this row until it coincides with the main column headed 9. The number at the intersection is 6.295.

Since in log 1.7990, 1 is a characteristic and 7990 is the mantissa then, the antilog of 1.7990 is $6.295 \times 10^1 = 62.95$.

In general, the anti – logarithm of a number x is denoted as $\log^{-1} x$.

Also, anti – logarithmic tables can be used to find a number whose logarithm is 0.5514 as follows:

In the extreme left hand column of the anti – logarithmic Table 4.2, locate .55.

Table 4.2. Anti-logarithms

10^x																	
x	0	1	...	6	7	8	9	Mean Differences (Add)									
								1	2	3	4	5	6	7	8	9
.00	1.000	1.002		1.014	1.016	1.019	1.021	0	0	1	1	1	1	2	2	2
.01	1.023	1.026		1.038	1.040	1.042	1.045	0	0	1	1	1	1	2	2	2
. . .																
.54	3.467	3.475		3.516	3.524	3.532	3.540	1	2	2	3	4	5	6	6	7
.55	3.548	3.556		3.598	3.606	3.614	3.622	1	2	2	3	4	5	6	7	7
.56	3.631	3.639		3.681	3.690	3.698	3.707	1	2	3	3	4	5	6	7	8
. . .																
.78	6.026	6.040		6.109	6.124	6.138	6.152	1	3	4	6	7	8	10	11	13
.79	6.166	6.180		6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
. . .																

The row with .55 meets the column labeled 1 in the main column part at 3.556. Continuation with this row (horizontally) meets a column labeled 4 in the difference column at number 3. This number 3 is added to the last digit of 3.556 to obtain 3.559.

Since the required number lies between 1 and 10, then the required number is 3.559. On rounding it to 2 decimal places, we get 3.56.

Therefore, log 3.56 = 0.5514. The number 3.56 is known as the anti – logarithm (or antilog) of 0.5514.

Example 4.19

Use mathematical tables to find the numbers whose logarithms are:

(a) 1.8810 (b) 3.7515 (c) 1.2466

Solution

(a) Let, $\log y = 1.8810$ (logarithm)

$y = \log^{-1}(1.8810)$ (antilogarithm)

$y = 7.603 \times 10^1$ (standard form)

$y = 76.03$

Therefore, 1.8810 is the logarithm of 76.03.

(b) Let, $\log x = 3.7515$ (logarithm)

$x = \log^{-1}(3.7515)$ (antilogarithm)

$x = 5.642 \times 10^3$ (standard form)

$x = 5642$

Therefore, 3.7515 is the logarithm of 5642.

(c) Let, $\log a = 1.2466$ (Logarithm)

$a = \log^{-1}(1.2466)$ (antilogarithm)

$a = 1.764 \times 10^1$ (standard form)

$a = 17.64$

Therefore, 1.2466 is the logarithm of 17.64.

Exercise 4.4

Answer the following questions:

1. Use the table of common logarithms to find the logarithms of each of the following numbers:

(a)	1.583	(b)	2.98	(c)	4.088	(d)	8.541
(e)	9.007	(f)	6	(g)	3.444	(h)	7.5348

2. Write the mantissa and the characteristic of each of the following logarithms:

(a)	0.0423	(b)	5.8181	(c)	2.6239	(d)	8.4894

3. Use mathematical tables to find the logarithm of each of the following numbers:

(a)	8725	(b)	700.1	(c)	76 408	(d)	4 300 000
(e)	83.7	(f)	111 327	(g)	20	(h)	1986
(i)	78085.9	(j)	354	(k)	43.657	(l)	493 000

4. Use mathematical tables to find the number whose logarithm is:

(a)	2	(b)	0.73	(c)	1.4533	(d)	5.9899
(e)	1.0043	(f)	0.7783	(g)	4.608	(h)	2.9
(i)	3.6721	(j)	2.8567	(k)	4.6755	(l)	0.6842

5. Determine the mantissa and characteristics in each of the following logarithms:

(a) log 4 127 (b) log 78.29

(c) log 2.828 (d) log 100 000 000

Logarithms of numbers greater than 0 but less than 1

The logarithm of a number which lies between 0 and 1, can be found by expressing the number in the standard form and using the laws of exponents.

Consider finding the logarithm of 0.00347

$$\log 0.00347 = \log\left(3.47\times10^{-3}\right)$$
$$= \log 3.47 + \log 10^{-3}$$
$$= \log 3.47 + (-3)$$

From the table, log 3.47= 0.5403.

The mantissa is 5403 and the characteristic is 0.

It should be noted that for the logarithm of a number which lies between 0 and 1, only the characteristic is negative. This is shown by putting a bar over the characteristic number. For example, $\log 0.00347 = \bar{3}.5403$ ($\bar{3}$ is read as 'bar three').

Note that, $\bar{3}.5403$ and -3.5403 are different numbers because, $\bar{3}.5403$ is not a decimal number. It is a combination of two numbers $\bar{3}$ and 5403, where $\bar{3}$ is the characteristic and 5403 is the mantissa. These numbers are independent, which means that, 5403 is not a decimal part of $\bar{3}$. But $\bar{3}$ represents the exponent of 10. In general, the characteristic can be positive or negative, while the mantissa is always positive. The number -3.5403 is a decimal number which consists of two parts, an integer part and a decimal part.

Basic operations with logarithms

Operations of addition, subtraction, multiplication and division with logarithms can be done as illustrated in the following examples:

Addition of logarithms

Example 4.20

Find $\bar{1}.5371 + 2.2436$.

Solution

Logarithm	Working	Result
$\bar{1}.5371$	$\bar{1} + 0.5371$	$\bar{1}.5371$
$+2.2436$	$+\ 2 + 0.2436$	$+\ 2.2436$
	$1 + 0.7807$	1.7807

Example 4.21

Find:

$\bar{3}.7379$
$+\ \bar{1}.3436$

Solution

Logarithm	Working	Result
$\bar{3}.7379$	$\bar{3}+0.7379$	$\bar{3}.7379$
$+\ \bar{1}.3436$	$+\ \bar{1}+0.3436$	$+\ \bar{1}.3436$
	$\bar{4}+1.0815$	$\bar{3}.0815$

Subtraction with logarithms

Example 4.22

Find:

2.5463
$-\bar{1}.3236$

Solution

Logarithm	Working	Result
2.5463	$2+0.5463$	2.5463
$-\bar{1}.3236$	$-(\bar{1}+0.3236)$	$-\bar{1}.3236$
	$2-(-1)+0.2227$	3.2227

Example 4.23

Find

$\bar{2}.5466$
$-\bar{1}.3796$

Solution

Logarithm	Working	Result
$\bar{2}.5466$	$\bar{2}+0.5466$	$\bar{2}.5466$
-1.3796	$-(1+0.3796)$	-1.3796
	$-3+0.1670$	$\bar{3}.1670$

Example 4.24

Find

$$\begin{array}{r} \bar{1}.5863 \\ -\bar{2}.3941 \\ \hline \end{array}$$

Solution

Logarithm	Working	Result
$\bar{1}.5863$	$\bar{1}+0.5863$	$\bar{1}.5863$
$-\bar{2}.3941$	$-\ (\bar{2}+0.3941)$	$-\bar{2}.3941$
	$\bar{1}-\bar{2}+0.1922$	1.1922

Example 4.25

Find:

$$\begin{array}{r} 0.5371 \\ -2.7436 \\ \hline \end{array}$$

Solution

Logarithm	Working	Result
0.5371	$\bar{1}+1.5371$	0.5371
-2.7436	$-\ (2+0.7436)$	-2.7436
	$\bar{1}-2+0.7935$	$\bar{3}.7935$

Exercise 4.5

Answer the following questions:

1. Find the sum of each of the following logarithms:

(a) $0.5371 + \bar{2}.7436$

(b) $\bar{1}.7391 + \bar{2}.6426$

(c) $1.7391 + \bar{2}.6426$

(d) $2.3413 + \bar{2}.6738$

(e) $\bar{4}.8871 + 0.4531$

(f) $1.7391 + \bar{2}.6426$

2. Find the difference of each of the following logarithms:

(a) $\bar{2}.7301 - 1.4496$

(b) $0.7301 - 2.5196$

(c) $4.3700 - 2.6455$

(d) $3.1771 - 2.2942$

(e) $\bar{4}.0171 - 0.4531$

(f) $\bar{2}.4391 - \bar{3}.6426$

Multiplication with logarithms

Example 4.26

Find the product of each of the following:

(a) 1.7301×3

(b) $\bar{2}.2721 \times 2$

(c) $\bar{3}.9324 \times 2$

Solution

(a) $1.7301 \times 3 = 5.1903$

(b) $\bar{2}.2721 \times 2 = \bar{4}.5442$

(c) $\bar{3}.9324 \times 2 = \bar{5}.8648$

Division with logarithms

If a dividend has no negative characteristic, division is done in the same way as for ordinary decimal numbers. But, when the dividend has a negative characteristic, increase the number under the bar to the next integer exactly divisible by the divisor. Then, balance the mantissa part by adding the appropriate whole number to the decimal part and divide.

Example 4.27

Find the quotient of each of the following logarithms:

(a) $\frac{1.8934}{2}$ (b) $\frac{\bar{5}.6812}{7}$ (c) $\frac{\bar{7}.2134}{5}$

Solution

(a) $\frac{1.8934}{2} = 0.9467$

(b) $\frac{\bar{5}.6812}{7} = \frac{\bar{7}+2.6812}{7}$
$= \bar{1} + 0.3830$
$= \bar{1}.3830$

(c) $\frac{\bar{7}.2134}{5} = \frac{\overline{10}+3.2134}{5}$
$= \bar{2} + 0.64268$
$= \bar{2}.6427$

Example 4.28

Given that $\log x = \bar{4}.7321$, determine $\log \sqrt[3]{x}$.

Solution

$\log \sqrt[3]{x} = \frac{1}{3} \log x$

$= \frac{\bar{4}.7321}{3}$

$= \frac{\bar{6}+2.7321}{3}$

$= \bar{2}.9107$

Therefore, $\log \sqrt[3]{x} = \bar{2}.9107$.

Exercise 4.6

Answer the following questions:

1. Use mathematical tables to find the logarithms of each of the following numbers:

   (a) 0.682 (b) 0.008 (c) 0.74 (d) 0.0000449

   (e) 0.031199 (f) 0.01478 (g) 0.125 (h) 0.00981

2. Use mathematical tables to find decimal numbers whose logarithms are:

   (a) 2.7050 (b) 1.001 (c) 6.006 (d) 1.3614

   (e) 2.4000 (f) 3.9600 (g) 5.8282 (h) 1.2009

   (i) 4.7711 (j) 4.8486 (k) 3.8525 (l) 1.4310

3. Determine the characteristic and the mantissa in each of the following:

   (a) log 0.0209 (b) log 0.61825

   (c) log 0.00000752 (d) log 0.07088

4. Find the sum of each of the following logarithms:

   (a) $2.3413 + \bar{1}.3876$ (b) $3.4364 + \bar{2}.7153$ (c) $0.8453 + \bar{3}.6738$

5. Find the difference of the following logarithms:

   (a) $1.7943 - \bar{2}.6061$ (b) $\bar{2}.6173 - 1.4291$ (c) $\bar{2}.3416 - 0.9874$

6. Find the product of the following logarithms:

   (a) $\bar{2}.7212 \times 3$ (b) $\bar{1}.4818 \times 4$ (c) $\bar{6}.0089 \times 2$

7. Find the quotient of each of the following logarithms:

   (a) $\bar{5}.8888 \div 4$ (b) $\bar{1}.4409 \div 3$ (c) $\bar{3}.7727 \div 2$ (d) $5.1343 \div 7$

8. If $\log x = 3.1425$, determine $\log \sqrt{x}$

9. If $\log x = 2.5543$, determine $\log \sqrt[4]{x}$.

10. If $\log x = 4.3217$, find the value of $\log \sqrt[3]{x}$.

11. Find x in each of the following:

    (a) $\log x = 2.9285$ (b) $\log x = 1.6990$

Calculations using logarithms

Logarithms can be used to evaluate products, quotients, powers and roots of numbers. To multiply numbers, first obtain their logarithms, add them and then find the antilogarithm of the sum.

Example 4.29

Find the product of 3.62 and 24.2 by using logarithmic tables.

Solution

$3.62 \times 24.2 =$

Number	Standard form	Logarithm
3.62	3.62×10^0	0.5587 } +
24.2	2.42×10^1	1.3838
		1.9425

Use mathematical tables to find the anti–logarithm of 0.9425, then multiply the answer by 10^1 to obtain 8.76×10^1. Therefore, $3.62 \times 24.2 = 87.6$.

Example 4.30

Find the product of 0.000056, 5279 and 0.35 by using logarithm tables.

Solution

$0.000056 \times 5279 \times 0.35 =$

Number	Standard form	Logarithm
0.000056	5.6×10^{-5}	$\bar{5}.7324$
5279	5.279×10^{3}	3.7225 (+)
0.35	3.5×10^{-1}	$\bar{1}.5441$
		$\bar{2}.9990$

Use mathematical tables to find the anti-logarithm of 0.9990, then multiply the answer by 10^{-2} to obtain 9.977×10^{-2}.

Therefore, $0.000056 \times 5279 \times 0.35 = 0.09977$.

Example 4.31

Find the value of $3.25 \div 0.071239$

Solution

Number	Standard form	Logarithm
3.25	3.5×10^{0}	0.5119 (−)
0.071239	7.1239×10^{-2}	$\bar{2}.8527$
		1.6592

Use mathematical tables to find the anti–logarithm of 0.6592, then multiply the answer by 10^{1} that is 4.562×10^{1}. Therefore, $3.25 \div 0.071239 = 45.62$.

Example 4.32

Evaluate $(0.5816)^3$.

Solution

Remember that: $\log a^x = x \log a$.

Number	Standard form	Logarithm
0.5816	5.816×10^{-1}	$\bar{1}.7646$
$(0.5816)^3$	$\longrightarrow$	$\bar{1}.7646$ $\times \quad 3$ $\bar{1}.2938$

Use mathematical tables to read the anti-logarithm of 0.2938, then multiply the answer by 10^{-1} that is 1.967×10^{-1}.

Therefore, $(0.5816)^3 = 0.1967$.

Example 4.33

Evaluate $\sqrt{8281}$

Solution

$\sqrt{8281} = (8281)^{\frac{1}{2}}$

Number	Standard form	Logarithm
8281	8.281×10^3	3.9181
$(8281)^{\frac{1}{2}}$		$3.9181 \div 2 = 1.9591$

Use mathematical tables to read the anti-logarithm of 0.9591, then multiply the answer by 10^1 that is 9.101×10^1. Therefore, $\sqrt{8281} = 91.01$.

Exercise 4.7

Use logarithms to calculate each of the following expressions:

1. 37.51×584
2. $52.6 \times 98451 \times 0.00324$
3. 3.72×0.064808
4. 0.000895×243
5. 0.01193×0.707
6. $288\,128\,000 \div 723.5$
7. $0.8 \div 0.00914$
8. $1.789 \div 885.3$
9. $549 \div 0.00907$
10. $6.12 \div 122.4$
11. $(0.0291)^5$
12. $(39.07)^2$
13. $(3.142)^3$
14. $(1.0757)^{10}$
15. $(0.007809)^2$
16. $\sqrt{3125}$
17. $(721.3)^3$
18. $(9\,380\,600)^{\frac{1}{3}}$
19. $\sqrt[4]{84.67}$
20. $\sqrt{0.00229}$

More calculations using logarithms

Calculations involving expressions with mixed operations are given in the following examples:

Example 4.34

Calculate the value of $\frac{9804 \times 23.19}{0.086 \times 41750}$.

Solution

	Number	Standard form	Logarithm
Numerator	9804	9.804×10^3	3.9914
	23.19	2.319×10^1	1.3653 (+)
			5.3567
Denominator	0.086	8.6×10^{-2}	$\bar{2}.9345$
	41750	4.1750×10^4	4.6207 (+)
			3.5552
Logarithm of a numerator			5.3567
Logarithm of a denominator			3.5552 (−)
Logarithm of a quotient			1.8015

Use mathematical tables to read the anti–logarithm of 0.1815, then multiply the answer by 10^1 that is 6.331×10^1.

Therefore, $\dfrac{9\,804 \times 23.19}{0.086 \times 41\,750} = 63.31$.

Example 4.35

Simplify $\dfrac{(38.1)^2 \times 0.005678}{\sqrt[3]{862}}$.

Solution

Number	Standard form	Logarithm
$(38.1)^2$	$(3.81 \times 10^1)^2$	1.5809 × 2 = 3.1618
		3.1618
0.005678	5.678×10^{-3}	$\bar{3}.7542$ (+)
Logarithm of a numerator		0.9160
862	8.62×10^2	2.9355
$(862)^{\frac{1}{3}}$		$2.9355 \div 3 = 0.9785$
Logarithm of a denominator		0.9785
Logarithm of a numerator		0.9160
Logarithm of a denominator		0.9785 (−)
Logarithm of a quotient		$\bar{1}.9375$

Use mathematical tables to read the anti – logarithm of 0.9375, then multiply the answer by 10^{-1} that is 8.66×10^{-1}.

Therefore, $\dfrac{(38.1)^2 \times 0.005678}{\sqrt[3]{862}} = 0.8660$.

Example 4.36

The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. Use logarithm tables to find the volume of a sphere with radius 20.06 cm (take $\pi = 3.142$).

Solution

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3} \times 3.142 \times (20.06)^3$$

$$V = \frac{4 \times 3.142 \times (20.06)^3}{3}$$

Number	Standard form	Logarithm
4	4×10^0	0.6021
3.142	3.142×10^0	0.4972 (+)
$(20.06)^3$	$(2.006 \times 10^1)^3$	1.3023×3
		5.0062
Logarithm of a numerator		
3	3×10^0	0.4771
Logarithm of a denominator		0.4771
Logarithm of a numerator		5.0062
Logarithm of a denominator		0.4771 (−)
Logarithm of a quotient		4.5291

Use mathematical tables to read the anti–logarithm of 0.5291, then multiply the answer by 10^4 that is 3.382×10^4. Therefore, the volume of the sphere is 33 820 cm^3.

Exercise 4.8

Answer the following questions:

In questions 1 to 10, evaluate each of the following expressions using logarithm tables.

1. $\frac{43.2 \times 0.0596}{0.00797}$

2. $\frac{3277 \times 0.097}{549}$

3. $\sqrt[3]{\frac{(43.93)^2}{14.4}}$

4. $\frac{723 \times 63.72}{0.21 \times 723520}$

5. $\frac{(0.93)^2}{0.088 \times 21.7}$

6. $\sqrt{\frac{62.85}{8.647 \times 3.204}}$

7. $\frac{876^3 \times 0.0537}{\sqrt{0.0009805}}$

8. $(3.35)^{10} + \left(\frac{(403.9)^2}{7.692}\right)^3$

9. $\sqrt[3]{\frac{318 \times 434}{17200}}$

10. $\frac{\left(\sqrt{0.83}\right) \times (713.4)}{142.5}$

11. Use logarithm table to calculate to 3 significant figures the value of R, if $R = \frac{0.000402 \times 286}{0.95}$.

12. The volume of a cylinder is given by $V = \pi r^2 h$. Find V if $r = 5.3$ cm and $h = 7.85$ cm (take $\pi = 3.142$).

13. If $d = \sqrt{\frac{2rh}{100}}$, find d when $r = 6370$ and $h = 265$.

14. Find the value of p if $\frac{p \times 155}{100} = \frac{85 \times 126}{298}$.

15. Taking π as 3.142, compute $2\pi\sqrt{\frac{9.81}{1.83}}$.

16. Calculate the radius of a sphere whose volume is 37.648 cm^3 ($V = \frac{4}{3}\pi r^3$, $\pi = 3.142$).

17. A rectangular box contains 256.8 cm^3 of air. If the length of the box is 7.35 cm and the width is 4.83 cm, find its height.

18. Given that $t = 49\sqrt{\frac{0.4597}{981 \times 0.76}}$ use logarithm tables to find the value of t.

19. The volume of a cone is given by $v = \frac{1}{3}\pi r^2 h$. Determine its volume, given that $r = 3.2$ cm and $h = 26.54$ cm (Take $\pi = 3.142$).

20. Find the number corresponding to each of the following logarithms to base 10.

(a) $0.3614 + (-1)$ (b) $0.4913 + (-2)$

(c) $0.4000 + (-2)$ (d) 3.6598

21. Find the logarithm of each of the following numbers (to base 10)

(a) 0.0412 (b) 7.01

(c) 14.3 (d) 0.000449

Chapter summary

1. A number in standard form is written as $A \times 10^n$ where $1 \leq A < 10$ and n is a positive or negative whole number.

2. A number in exponential form can be expressed in logarithmic form.

For example:

Exponential form	Logarithmic form
$1000 = 10^3$	$\log_{10} 1000 = 3$
$1 = 10^0$	$\log_{10} 1 = 0$
$0.01 = 10^{-2}$	$\log_{10} 0.01 = -2$

Generally, if $x = a^b$, then $\log_a x = b$.

3. The laws of logarithms are:

   Logarithm of a product: $\log_a(MN) = \log_a M + \log_a N$

   Logarithm of a quotient: $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

   Logarithm of a power: $\log_a (M)^p = p\log_a M$

   Logarithm of identical power and base: $\log_a a = 1$

   Logarithm of a root: $\log_a \sqrt[m]{x^n} = \frac{n}{m}\log_a x$

4. The logarithm of a number is written in two parts namely, the characteristic and the mantissa.

5. The characteristic of a logarithm may be found by expressing the number in standard form, $A \times 10^n$, where n is the required characteristic (either positive, negative or 0). The mantissa of a logarithm is found in mathematical tables.

7. For logarithms of numbers greater than 0 and less than 1, use bar notation to represent the characteristic. For example, the logarithm of 0.086 is $\bar{2}.9345$.

8. The principles of calculation using logarithms depend on the laws of exponents. That is;
   (i) when multiplying, add logarithms;
   (ii) when dividing, subtract; and
   (iii) when raising to a power, multiply by the exponent.

Revision exercise 4

Answer the following questions:

1. Write each of the following numbers in standard form:

   (a) 8 419 000 (b) 45.7 (c) 716
   (d) 0.000123 (e) 4 (f) 0.005

2. Determine the decimal numerals for each of the following expressions:

   (a) 9.15×10^5 (b) 8×10^{-3}
   (c) 1.06×10^2 (d) 2.5×10^1

3. Compute each of the following expressions, giving your answers in standard form:

   (a) $\left(8\times10^{-5}\right)\times\left(27.5\times10^{15}\right)$ (b) $\left(12.5\times10^{4}\right)\times\left(8\times10^{-7}\right)$

   (c) $\dfrac{8\times10^{-3}}{5\times10^{-5}}$ (d) $\dfrac{1.728\times10^{7}}{1.2\times10^{2}}$

4. Given the formula $Q=\dfrac{V^2}{R}$, use mathematical tables to calculate Q when:

   (a) $R=5\times10^{3}, V=2\times10^{-1}$ (b) $R=4\times10^{2}, V=2\times10^{2}$.

5. Find the value of x in each of the following equations:

   (a) $\log_8 x=4$ (b) $\log_x\left(\dfrac{1}{125}\right)=-3$ (c) $\log x=3$

6. Determine the value of x in each of the following equations:

   (a) $\log(x^2+3x-44)=1$ (b) $\log(2x+1)=0$

7. Determine the number whose logarithm in:

   (a) base 10 is 6 (b) base 6 is 6.

8. Given that $\log x = 8.0524$, find $\log\sqrt[3]{x}$.

9. Given that $\log 2 = 0.30103$, $\log 5 = 0.69897$, and $\log 7 = 0.84510$, calculate without using mathematical tables the value of $\log\left(\frac{35}{2}\right)$.

10. Without using mathematical tables, determine the value of $\log 5 - \log 8 + 4\log 2$.

11. Simplify each of the following without using mathematical tables or calculators:

    (a) $\frac{\log\sqrt{10}}{\log 10} \times \log 100$ (b) $\log_4 176 - \log_4 11$

12. Use mathematical tables to find the logarithm of each of the following numbers:

    (a) 0.8008 (b) 724 079 (c) 0.0002 (d) 23.9

13. Find the value of x if:

    (a) $\log x = 4.3217$ (b) $\log x = 2.5543$

14. If $3^x = 8$ find the value of x.

15. By using mathematical tables evaluate the following expressions:

    (a) $40.5 \times 300 \times 0.008904$ (b) 0.632×3.456

16. Use mathematical tables to determine the value of:

    (a) $(2.09)^{10}$ (b) $(0.5216)^{\frac{2}{3}}$

17. Compute each of the following expressions by using mathematical tables:

    (a) $\frac{8.802 \times 0.00123}{0.01252 \times 352080}$ (b) $\frac{\sqrt{(2.16)^3}}{0.534 \times 3333}$

18. In two concentric circles, if R is the radius of the larger circle and r the radius of the smaller circle, the area of the ring between the two circles is given by $A = \pi(R^2 - r^2)$ mm^2. Use mathematical tables to find the area of the ring if $R = 12.05$ mm and $r = 10.05$ mm. (Take $\pi = 3.142$).

19. Use mathematical tables to calculate the value of T from the formula $T = 2\pi\sqrt{\frac{l}{g}}$, given $l = 0.825$ and $g = 9.81$ (use $\pi = 3.142$).

20. Given that $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ use logarithm tables to find u if $v = 47.9$ cm and $f = 10.28$ cm.

21. By using mathematical tables and the formula $v^2 = u^2 + 2as$, calculate the value of s if it is known that $u = 18.5$, $v = 36$, and $a = 3.8$.

22. The formula for finding the volume of a right circular cylinder is given by $V = \pi r^2 h$. Use mathematical tables to find the value of r, given that $V = 64.91$ cm and $h = 3.907$ cm (use $\pi = 3.142$).

23. Given that, $\frac{4}{3}\pi r^3 = 234.5$, use mathematical tables to calculate $4\pi r^2$ (use $\pi = 3.142$).

24. Find the value of x, if $(\log_2 x)(-3 + \log_2 x) = 4$.

25. Solve for x, given that $\log_3 x + 3\log_x 3 = 4$.

Chapter Five

Congruence

Introduction

The objects and geometrical figures found in our surroundings can have the same shape and size or same shape but different sizes. Congruence is the concept used to describe an object and its mirror image. Two objects or figures are congruent if they are of the same shape and size. In geometry, shapes are congruent when they are exactly the same. In this chapter, you will learn about postulates, proofs and theorems on congruence as well as congruence of triangles. The competences developed in this chapter will enable you to determine distances without doing actual measurements. You will also be able to identify replacements of worn–out parts of machines.

Congruence of triangles

The term congruence is derived from the Latin word *congruentiam* which means, agree or fit together exactly. Two triangles are congruent if all their corresponding sides and angles are equal. However, it is necessary to find all six dimensions (3 sides and 3 angles) of each triangle. Thus, congruence of triangles can be determined by knowing three out of the six dimensions. The mathematical symbol for congruence is $\cong$.

Activity 5.1: Identifying congruent objects

Materials required: Rulers, textbooks, and mathematical sets.

Perform the activity by doing the following tasks in pairs or groups.

1. Match the following objects and observe if they fit exactly or not:
   (a) Two rulers of length 30 cm of the same design
   (b) Two Mathematics textbooks
   (c) Two mathematical sets
2. Mention other things or objects in your school which fit exactly.
3. Is it possible for two parallelograms to have corresponding sides equal but different corresponding angles? Draw the diagrams to support your answer.

Postulates, proofs and theorems

Postulates are basic statements of facts which are assumed to be true.

The following statements are examples of postulates:

(a) If two straight lines intersect, the sum of two adjacent angles is equal to 180 degrees.

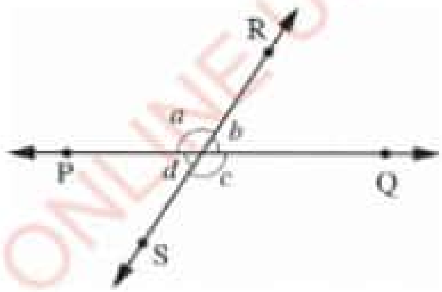


Figure 5.1: *Intersecting lines* $\overline{PQ}$ and $\overline{RS}$ *which make angles* a, b, c and d.

Figure 5.1 shows that, the sum of any two adjacent angles a and b, b and c, c and d or d and a is equal to 180 degrees.

That is, $a + b = b + c = c + d = d + a = 180°$

(b) If two straight lines intersect, the vertically opposite angles formed have the same degree measure, therefore in Figure 5.1, $a=c$ and $b=d$.

A proof is an argument for a statement which shows that the stated assumptions logically imply the conclusion.

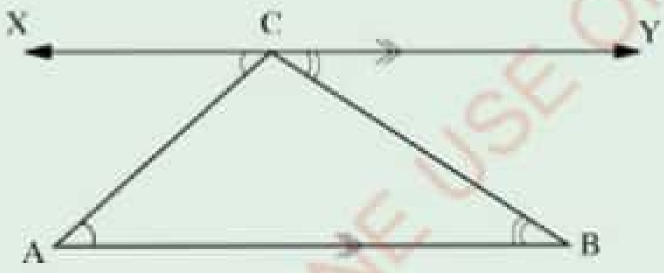
Theorems are important statements that have been proven to be true. For example, the statement: The sum of interior angles of a triangle is 180°, is a theorem.

Example 5.1

Prove that the sum of interior angles of a triangle is 180°.

Proof

Consider the following figure:



Given ΔABC, required to show that, $C\hat{A}B + B\hat{C}A + A\hat{B}C = 180°$.

Construction: Draw line $\overline{XY}$ through **C** parallel to $\overline{AB}$. Thus,

$A\hat{C}X = C\hat{A}B$ (alternate interior angles as $\overline{XY}//\overline{AB}$)

$Y\hat{C}B = A\hat{B}C$ (alternate interior angles as $\overline{XY}//\overline{AB}$)

Since $A\hat{C}X + B\hat{C}A + Y\hat{C}B = 180°$ (degree measure of a straight angle), then $C\hat{A}B + B\hat{C}A + A\hat{B}C = 180°$.

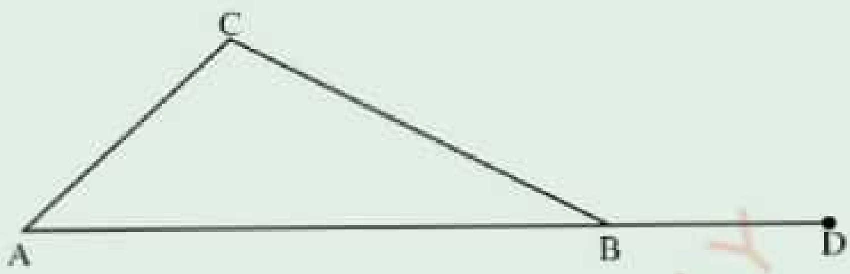
Therefore, the sum of interior angles of a triangle is 180°.

Example 5.2

Prove that the sum of two interior angles of a triangle is equal to the exterior angle of the third interior angle.

Proof

Consider the following figure.



Given ΔABC with $\overline{AB}$ extended to D, required to show that

$C\hat{A}B + B\hat{C}A = C\hat{B}D$

From $C\hat{A}B + B\hat{C}A + A\hat{B}C = 180^\circ$ (sum of interior angles of a triangle), and $A\hat{B}C + C\hat{B}D = 180^\circ$ (degree measure of a straight angle), it follow that;

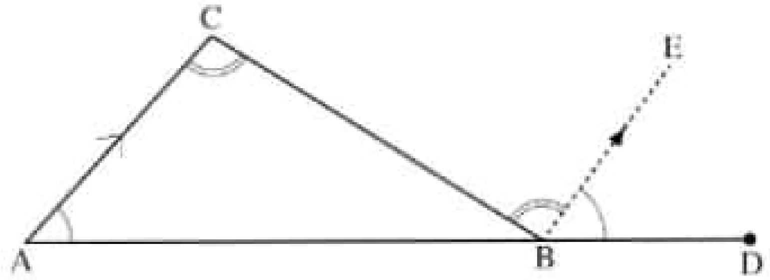
$C\hat{A}B + B\hat{C}A + A\hat{B}C = A\hat{B}C + C\hat{B}D$.

But $A\hat{B}C$ is common to both sides. Thus, $C\hat{A}B + B\hat{C}A = C\hat{B}D$.

Therefore, the sum of two interior angles of a triangle is equal to the exterior angle of the third interior angle.

Alternative proof

Consider the following figure.



Given ΔABC in which $\overline{AB}$ is extended to D.

It is required to prove that $C\hat{A}B + B\hat{C}A = C\hat{B}D$.

Construction: Draw $\overline{BE}$ parallel to $\overline{AC}$ through B.

It follows that,

$C\hat{A}B = E\hat{B}D$ (corresponding angles as $\overline{AC}//\overline{BE}$) and $B\hat{C}A = C\hat{B}E$ (alternate interior angles as $\overline{AC}//\overline{BE}$).

Thus, $C\hat{A}B + B\hat{C}A = E\hat{B}D + C\hat{B}E$.

But $E\hat{B}D + C\hat{B}E = C\hat{B}D$.

Hence, $C\hat{A}B + B\hat{C}A = C\hat{B}D$.

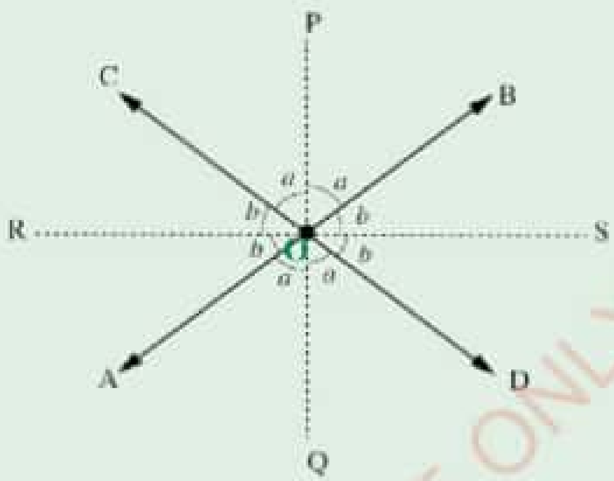
Therefore, the sum of two interior angles of a triangle is equal to the exterior angle of the third interior angle.

Example 5.3

Prove that the bisectors of the angles formed by two intersecting straight lines are at right angles to each other.

Proof

Consider the following figure.



Given two intersecting lines $\overline{AB}$ and $\overline{CD}$. Let O be the point where the lines $\overline{AB}$ and $\overline{CD}$ intersect such that the bisector $\overline{PQ}$ bisects the opposite angles $B\hat{O}C$ and $A\hat{O}D$ at equal angles a each, and the bisector $\overline{RS}$ bisects the opposite angles $A\hat{O}C$ and $B\hat{O}D$ at equal angles b each. Required to prove that, $a+b=90°$.

From the figure, it is noted that;

$a + b + b + a = 180°$ (degree measure of a straight angle)

That is $2a+2b=180°$ or $2(a+b)=180°$

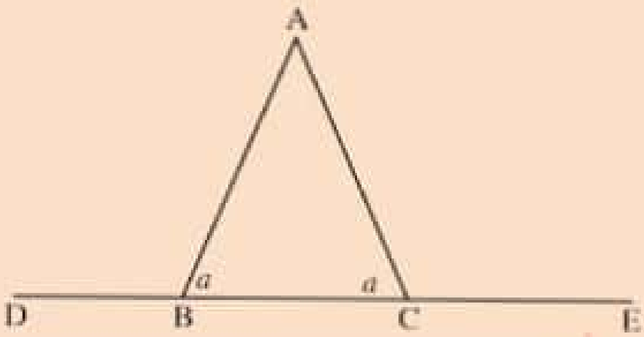
Thus, $a+b=90°$.

Therefore, the bisectors are at right angles to each other.

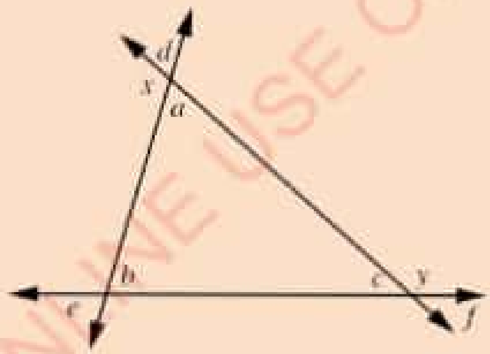
Exercise 5.1

Answer the following questions:

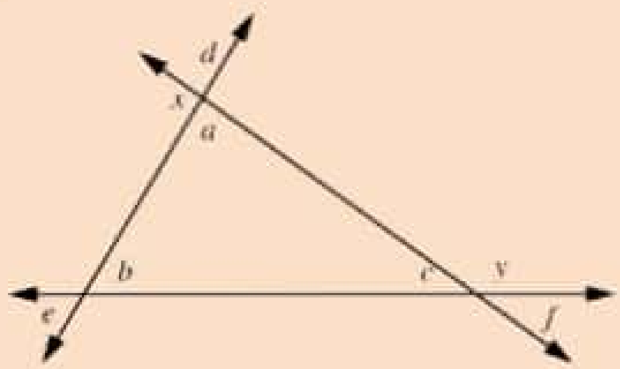
1. Use the following figure to prove that $A\hat{B}D = A\hat{C}E$.



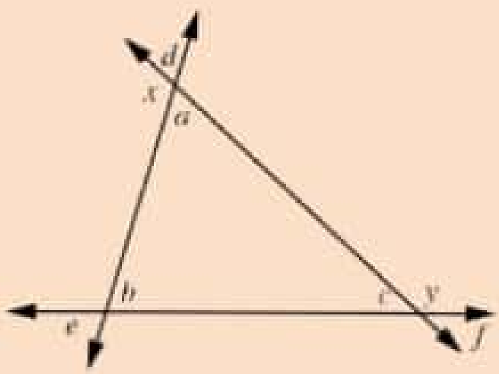
2. If $e = d$, then use the following figure to prove that $a = b$.



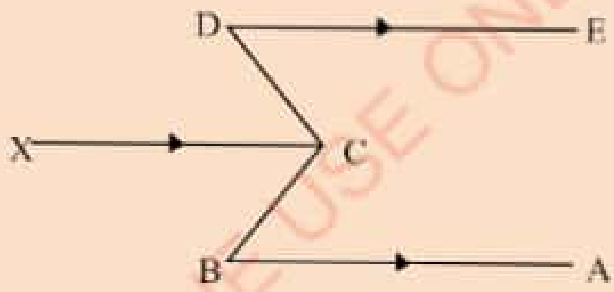
3. If $a = c$, then use the following figure to prove that $x = y$.



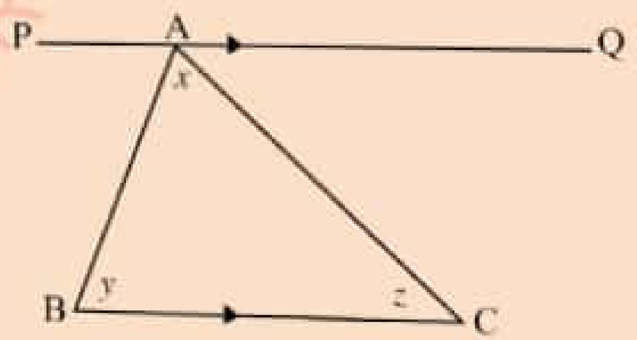
4. Use the following figure to prove that $a = c$, if $d + y = 180°$.



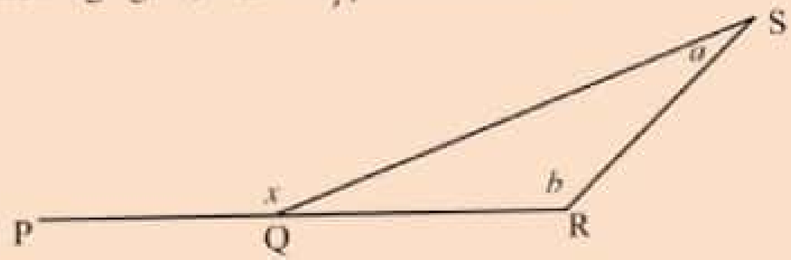
5. Use the following figure to prove that $B\hat{C}D = E\hat{D}C + C\hat{B}A$.



6. In the following figure, if $\overline{AC}$ bisects $B\hat{A}Q$, then prove that $x = z$.

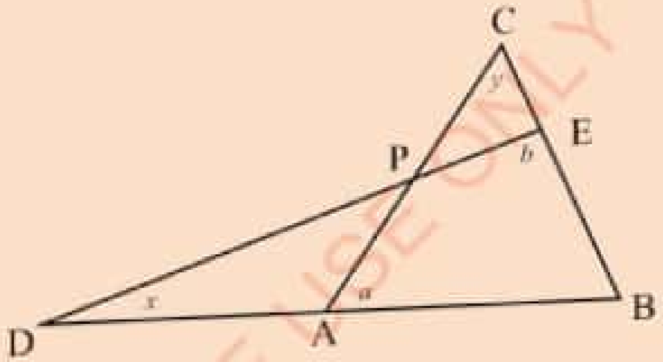


7. In the following figure, if $x = 2a$, prove that $a = b$.

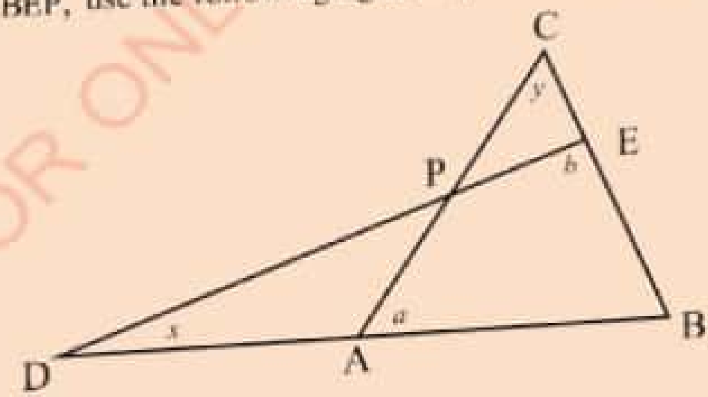


8. Prove that the sum of interior angles of a quadrilateral is equal to four right angles.

9. Use the following figure to prove that if $x = y$, then $a = b$.



10. If $C\hat{A}B = B\hat{E}P$, use the following figure to prove that $B\hat{D}E = A\hat{C}B$.



Postulates for congruence of triangles

Side – Side – Side (SSS) postulate

Two triangles are congruent if their corresponding sides are equal. Thus, each pair of the corresponding sides has equal length. Figure 5.2 illustrates the statement of this postulate.

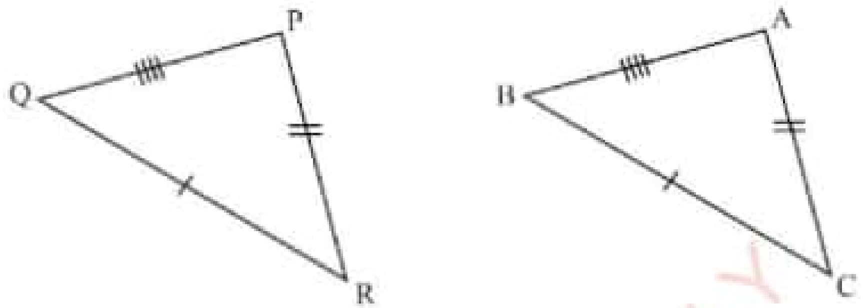


Figure 5.2: *Two congruent figures with SSS postulate.*

Figure 5.2 shows that,

$\overline{AB} = \overline{PQ}$ (given)

$\overline{BC} = \overline{QR}$ (given)

$\overline{AC} = \overline{PR}$ (given)

Since the sides of triangle ABC are equal to the corresponding sides of triangle PQR, then the two triangles are exactly the same.

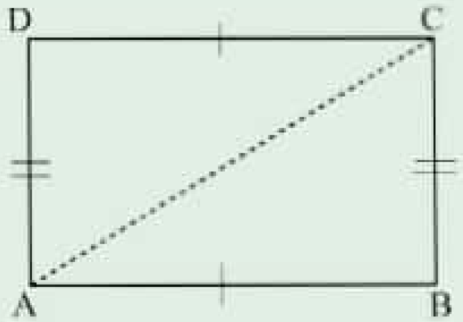
Therefore, it follows that, $\Delta ABC \cong \Delta PQR$ (by SSS).

SSS is the abbreviation of **Side–Side–Side.**

Since the two triangles are congruent, it follows that, their corresponding angles are equal in measures. That is, $C\hat{A}B = R\hat{P}Q$, $A\hat{B}C = P\hat{Q}R$ and $B\hat{C}A = Q\hat{R}P$

Example 5.4

Use the following figure to prove that $\Delta ABC \cong \Delta CDA$, and hence deduce that $D\hat{C}A = B\hat{A}C$.



Proof

From figure ABCD, $\overline{AB} = \overline{DC}$ and $\overline{AD} = \overline{BC}$.

Required to prove that $\Delta ABC \cong \Delta CDA$ and $D\hat{C}A = B\hat{A}C$.

Construction: Draw a line joining the points A and C.

ΔABC and ΔCDA, indicate that,

$\overline{AB} = \overline{DC}$ (given)

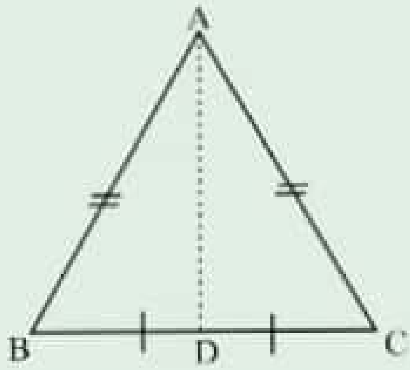
$\overline{BC} = \overline{AD}$ (given)

$\overline{AC}$ is a common side to both triangles

Therefore, $\Delta ABC \cong \Delta CDA$ (By SSS) and $B\hat{A}C = D\hat{C}A$ (definition of congruence).

Example 5.5

Triangle ABC is an isosceles triangle in which $\overline{AB}$ and $\overline{AC}$ are equal. If D is the midpoint of $\overline{BC}$, prove that $\Delta ABD \cong \Delta ACD$.



Proof

Given the ΔABC such that $\overline{AB} = \overline{AC}$ and D is the mid-point of $\overline{BC}$.
Required to prove that $\Delta ABD \cong \Delta ACD$.

Construction: Draw a line to join the points A and D.

From $\Delta ABD \cong \Delta ACD$ it follows that,

$\overline{AB} = \overline{AC}$ (given)

$\overline{BD} = \overline{DC}$ (given)

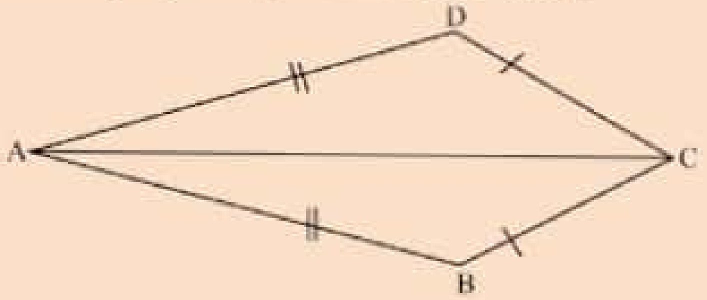
$\overline{AD}$ is a common side to both triangles.

Therefore, $\Delta ABD \cong \Delta ACD$ (by SSS).

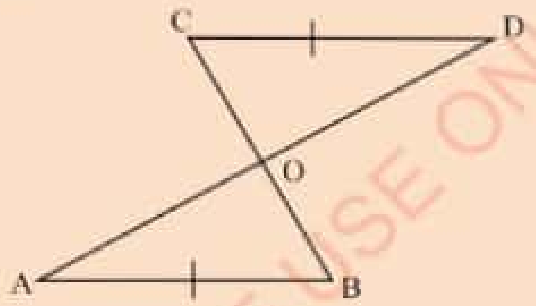
Exercise 5.2

Answer the following questions:

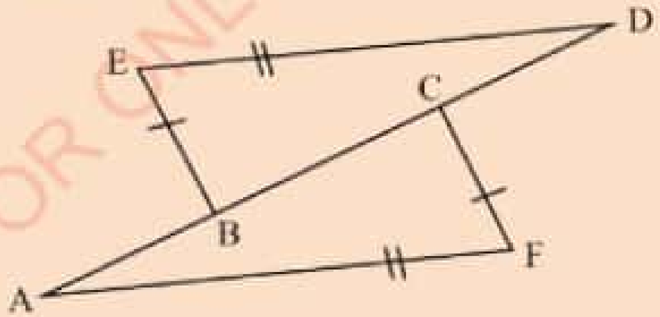
1. Use the following figure to prove that $A\hat{B}C = A\hat{D}C$.



2. In the following figure, $\overline{AD}$ and $\overline{BC}$ bisect each other at O. Prove that $\overline{AB}$ is parallel to $\overline{CD}$.



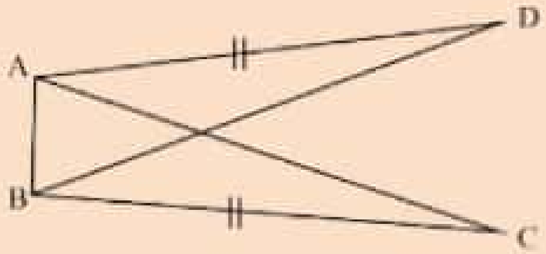
3. In the following figure, $\overline{AB} = \overline{CD}$ and ABCD is a straight line. Prove that $B\hat{A}F = C\hat{D}E$.



4. If OAB is a triangle in which $\overline{OA} = \overline{OB}$, and N is the mid-point of $\overline{AB}$. Prove that $O\hat{N}A = O\hat{N}B$.

5. If $\overline{AB}$ and $\overline{CD}$ are two equal parallel chords of a circle with centre O, prove that $A\hat{O}B = C\hat{O}D$.

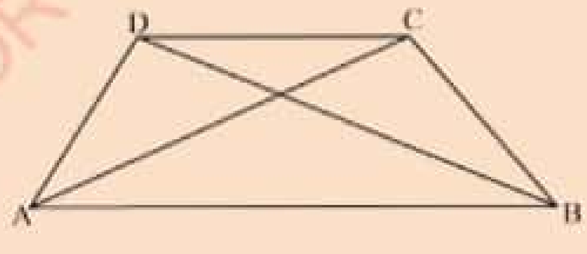
6. Use the following figure to prove that if $\overline{AC} = \overline{BD}$, then $A\hat{C}B = B\hat{D}A$.



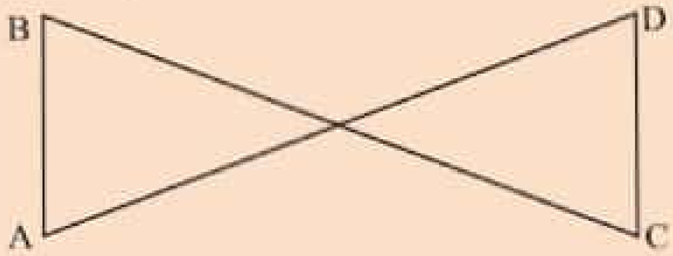
7. Two circles with centres at A and B intersect at points P and Q. Prove that $\overline{AB}$ bisects $P\hat{A}Q$ (Hint: use triangles APB and AQB).

8. In the following figure, $\overline{AD} = \overline{AB}$ and $\overline{CD} = \overline{CB}$, prove that $\overline{AC}$ bisects $D\hat{A}B$ and $D\hat{C}B$.

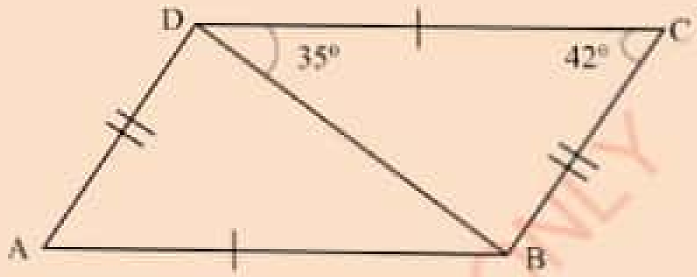
9. In the following figure, $\overline{AD} = \overline{BC}$ and $\overline{AC} = \overline{BD}$. Prove that $\Delta ABD \cong \Delta BAC$.



10. In the following figure $\overline{AD} = \overline{BC}$ and $\overline{AB} = \overline{CD}$, prove that $B\hat{A}D = D\hat{C}B$. (Hint: Join B and D)



11. Use the following figure to find the value of $A\hat{B}D$.



Side – Angle – Side (SAS) postulate

Two triangles are congruent if two pairs of their corresponding sides are such that each pair has equal length and the enclosed angles between given sides are equal. Figure 5.3 illustrates the statement of the postulate.

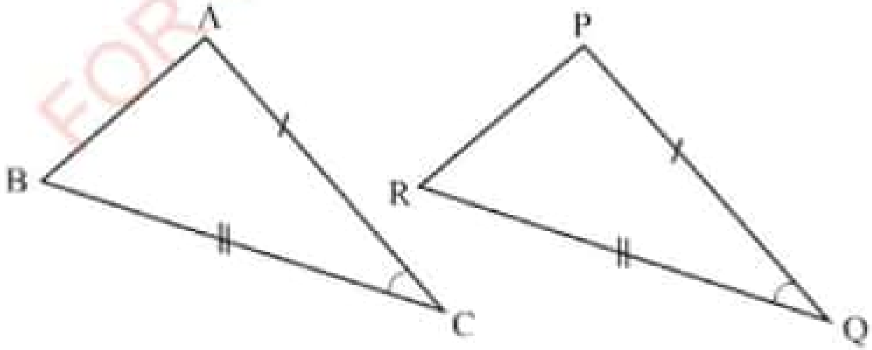


Figure 5.3: *Two congruent figures with SAS postulate*

Figure 5.3 shows that,

$\overline{AC} = \overline{PQ}$ (given)

$B\hat{C}A = R\hat{Q}P$ (given)

$\overline{CB} = \overline{RQ}$ (given)

If the two triangles satisfy the SAS conditions, the triangles will fit exactly.

Therefore, $\Delta ABC \cong \Delta PRQ$ (by SAS).

SAS is the abbreviation for **Side–Angle–Side**

Since the two triangles are congruent, it follows that, all the corresponding sides and angles are equal.

That is, $B\hat{A}C = Q\hat{P}R$, $\Delta ABC \cong \Delta PRQ$ and $\overline{AB} = \overline{PR}$.

Example 5.6

Use the following figure to prove that $\Delta ADC \cong \Delta CBA$.

Proof

From the quadrilateral ABCD, $\overline{AB} = \overline{DC}$ and

$D\hat{C}A = B\hat{A}C$ (alternate interior angles as $\overline{AB} \parallel \overline{DC}$)

Required to prove that $\Delta ADC \cong \Delta CBA$.

From ΔADC and ΔCBA it follows that,

$\overline{DC} = \overline{AB}$ (given)

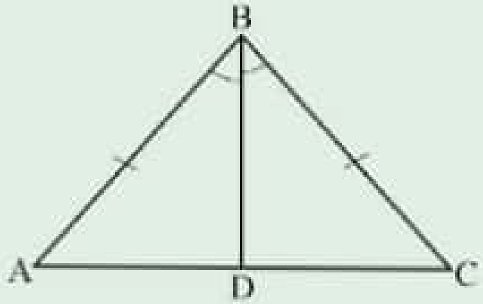
$D\hat{C}A = B\hat{A}C$ (given)

$\overline{AC}$ is a common side to both triangles.

Therefore, $\Delta ADC \cong \Delta CBA$. (by SAS).

Example 5.7

Use the following figure to prove that $\overline{AD} = \overline{CD}$.



Proof

Given ΔABC such that $\overline{BA} = \overline{BC}$ and $A\hat{B}D = D\hat{B}C$ required to prove that $\overline{AD} = \overline{DC}$.

From ΔABD and ΔCBD it follows that,

$\overline{BA} = \overline{BC}$ (given)

$A\hat{B}D = C\hat{B}D$ (given)

$\overline{BD}$ is a common side to both triangles.

Thus, $\Delta ABD \cong \Delta CBD$ (by SAS)

Therefore, $\overline{AD} = \overline{CD}$ (definition of congruence of triangles).

It is important to note that the angle must always be enclosed between the two equal sides. Otherwise, the triangles will not be congruent.

For example, if the triangle AB_1C in Figure 5.4 is such that $\overline{AC} = 6$ cm, $C\hat{A}B_1 = 30°$, and $\overline{CB_1} = \overline{CB_2} = 4$ cm.

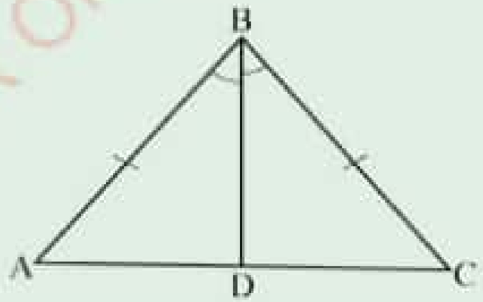


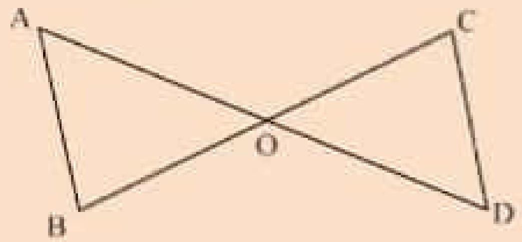
Figure 5.4: *Construction of non – congruent triangles*

In Figure 5.4, the triangles AB_1C and AB_2C satisfy the information given about the triangle, but they are not congruent.

Exercise 5.3

Answer the following questions:

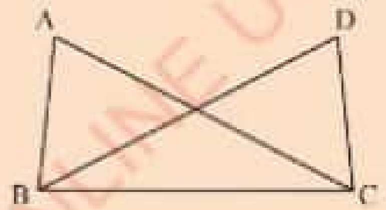
1. Given that in the following figure, $\overline{AO} = \overline{OD}$ and $\overline{OB} = \overline{OC}$.



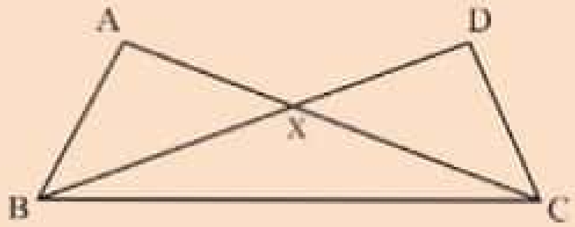
   (a) Prove that $\overline{AB} = \overline{CD}$

   (b) Write the angle which is equal to $B\hat{A}O$.

2. In the following figure, if $\overline{AB} = \overline{DC}$ and $A\hat{B}C = D\hat{C}B$, then prove that $\overline{AC} = \overline{DB}$.



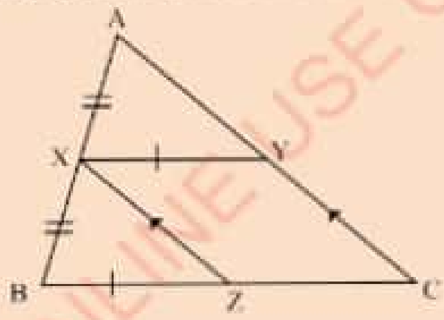
3. In the following figure, $\overline{AX} = \overline{DX}$ and $\overline{BX} = \overline{CX}$, prove that $B\hat{A}C = C\hat{D}B$.



4. Use the following figure to prove that $A\hat{D}B = B\hat{C}A$, given that $\overline{AC} = \overline{BD}$ and $B\hat{A}C = A\hat{B}D$.

5. Given a quadrilateral ABCD such that $\overline{AB} = \overline{DC}$ and $\overline{AC}$ bisects $D\hat{A}B$. Prove that $\overline{AD} = \overline{BC}$.

6. Use the following figure to prove that $\overline{XZ} = \overline{AY}$.



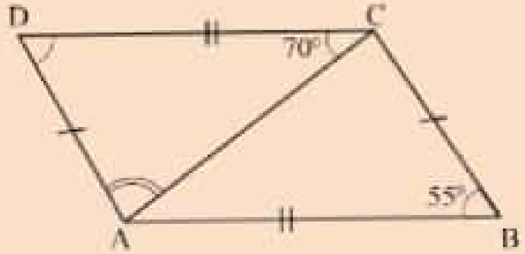
7. Given a quadrilateral ABCD with two line segments $\overline{DC}$ and $\overline{AB}$ are drawn apart such that $\overline{AB} = \overline{DC}$ and $A\hat{B}D = B\hat{D}C$. Prove that $D\hat{A}B = B\hat{C}D$.

8. A triangle ABC is given such that $\overline{AC} = \overline{BC}$ and $\overline{DC}$ is a bisector of $B\hat{C}A$. Prove that $A\hat{D}C = B\hat{D}C$.

9. If O is the centre of the circle ACB and $A\hat{O}C = C\hat{O}B$, prove that $\overline{AC} = \overline{CB}$.

10. The circle ABCD is centered at O. If $\overline{AC}$ and $\overline{BD}$ are diameters of the circle and line segments $\overline{AD}$, $\overline{AB}$ and $\overline{CB}$ are drawn, prove that $\overline{AD} = \overline{BC}$.

11. Find the value of $D\hat{A}C$ in the following figure.



Angle – Angle –Side (AAS) postulate

Two triangles are congruent if two pairs of corresponding angles are such that the angles in each pair are equal, and the lengths of a pair of corresponding sides are equal. **Angle – Angle – Side** postulate is abbreviated as AAS.

Figure 5.5 illustrates the statement of this postulate.

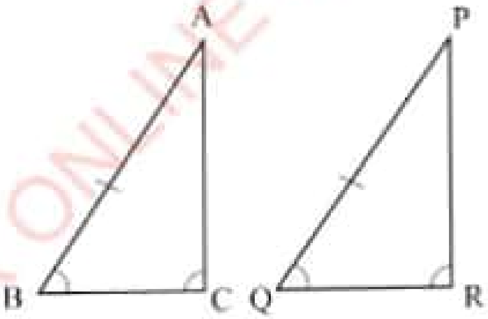


Figure 5.5: *Two congruent figures with AAS postulate*

Figure 5.5 shows that,

$\overline{BA} = \overline{QP}$ (given)

$A\hat{B}C = P\hat{Q}R$ (given)

$B\hat{C}A = Q\hat{R}P$ (given)

If two triangles satisfy the conditions stated in the AAS postulate, then the triangles are congruent.

Therefore, $\Delta ABC \cong \Delta PQR$ (by AAS).

Since the two triangles are congruent, it also follows that, all corresponding sides and angles are equal. That is,

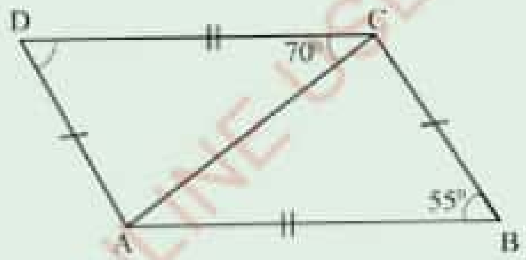
$\overline{BA} = \overline{QP}$

$\overline{AC} = \overline{PR}$

$B\hat{A}C = Q\hat{P}R$.

Example 5.8

In the following figure, prove that $\Delta ABC \cong \Delta CDA$.



Proof

Given a parallelogram ABCD with the diagonal $\overline{AC}$, required to prove that, $\Delta ABC \cong \Delta CDA$.

From ΔABC and ΔCDA it follows that,

$C\hat{A}B = A\hat{C}D$ (alternate interior angles as, $\overline{AB} // \overline{DC}$)

$A\hat{C}B = C\hat{A}D$ (alternate interior angles as, $\overline{AD} // \overline{BC}$)

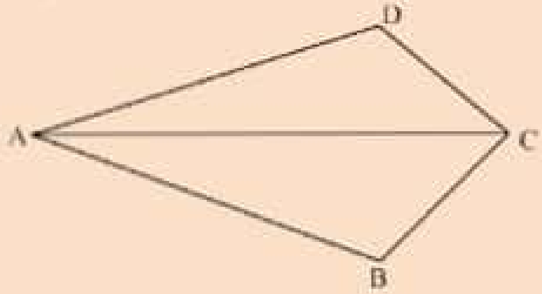
$\overline{AC}$ is a common side to both triangles.

Therefore, $\Delta ABC \cong \Delta CDA$ (by AAS).

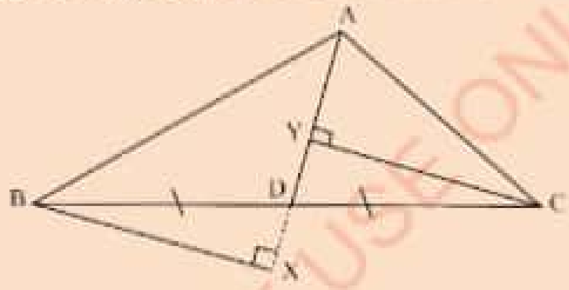
Exercise 5.4

Answer the following questions:

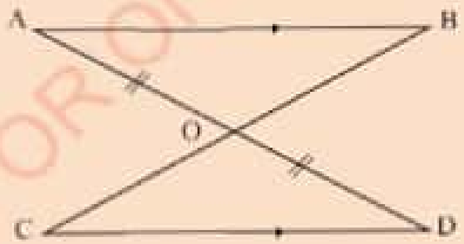
1. In the following figure, $\overline{AC}$ bisects $B\hat{A}D$ and $B\hat{C}D$, prove that $\overline{AB} = \overline{AD}$.



2. Given that X and Y are the feet of the perpendiculars from B and C to $\overline{AD}$ as shown in the following figure. Prove that $\overline{BX} = \overline{CY}$.



3. In the following figure, line segments $\overline{CB}$ and $\overline{AD}$ intersect at O such that $\overline{AO} = \overline{OD}$. If $\overline{AB}$ is parallel to $\overline{CD}$, prove that $\overline{AB} = \overline{CD}$.



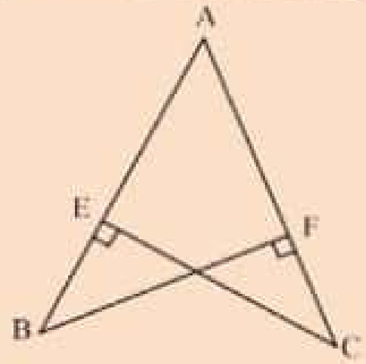
4. Use the figure in question 3 to prove that,

   (a) $\overline{AO} = \overline{OD}$

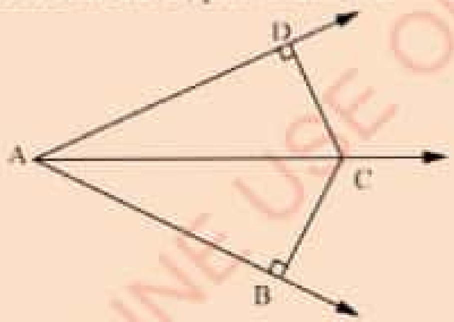
   (b) $\overline{CO} = \overline{OB}$

5. In ΔPQR, X is the midpoint of $\overline{PQ}$, Y and Z are the mid – points of $\overline{PR}$ and $\overline{QR}$, respectively. If XYRZ is a parallelogram, prove that $\overline{XY} = \overline{QZ}$.

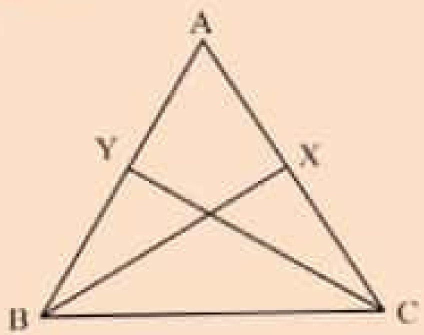
6. In the following figure, $\overline{AB} = \overline{AC}$. Prove that $\overline{BF} = \overline{CE}$.



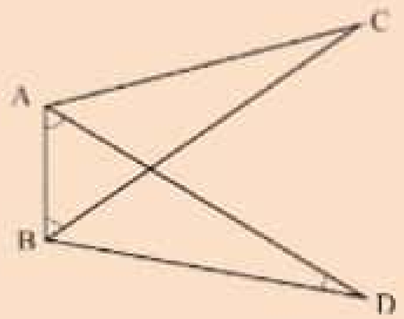
7. In the following figure, $\overline{AC}$ is the bisector of $B\hat{A}D$ and B and D are the feet of the perpendiculars from C, prove that $\overline{AB} = \overline{AD}$.



8. In the following figure, $A\hat{B}C = A\hat{C}B$, $\overline{BX}$ bisects $A\hat{B}C$ and bisects $A\hat{C}B$, prove that $\overline{BX} = \overline{CY}$.

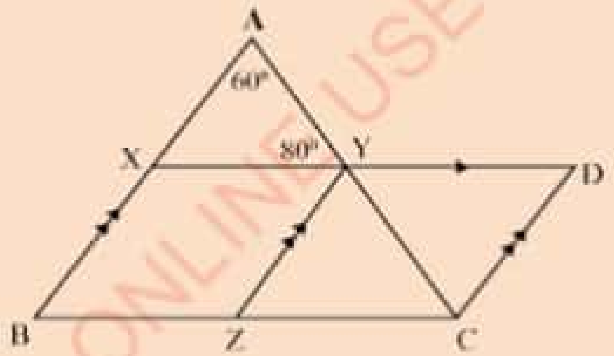


9. Use the following figure to prove that $\overline{BC} = \overline{AD}$, if $A\hat{B}C = B\hat{A}D$ and $B\hat{C}A = A\hat{D}B$.



10. Given a quadrilateral ABCD such that $\overline{AD}$ is parallel to $\overline{BC}$ and O is the point of intersection of the diagonals. If $\overline{AD} = \overline{BC}$, prove that $\Delta AOD \cong \Delta COB$.

11. Find the value of $Y\hat{Z}C$ in the following figure.



Right angle – Hypotenuse – Side (RHS) postulate

Two right – angled triangles are congruent if their hypotenuses and a pair of the corresponding sides have equal length. Figure 5.6 illustrstes the statement of this postulate.

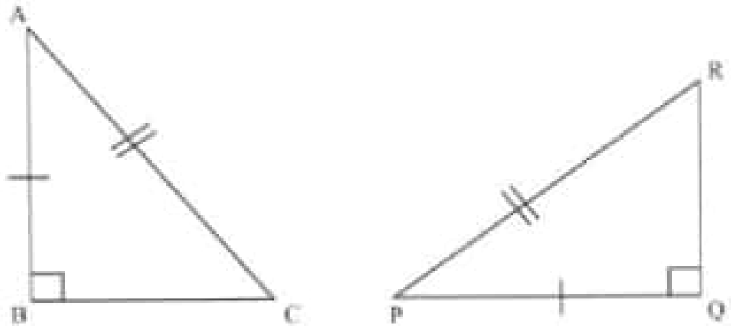


Figure 5.6: *Two congruent right – angled triangles with RHS postulate*

Figure 5.6 shows that,

$\overline{AB} = \overline{PQ}$ (given)

$\overline{AC} = \overline{PR}$ (given)

$A\hat{B}C = P\hat{Q}R = 90°$ (given).

If two triangles satisfy the conditions given in the RHS postulate, the triangles will fit each other exactly. Thus, Figure 5.6, shows that $\Delta ABC \cong \Delta PQR$ (by RHS).

RHS is the abbreviation for **Right angle – Hypotenuse – Side.**

Since the two triangles in Figure 5.6 are congruent, it also follows that, all its corresponding sides and angles are equal. That is,

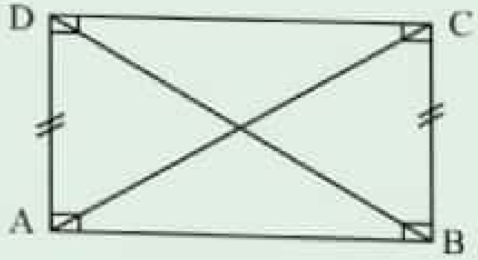
$\overline{BC} = \overline{QR}$

$B\hat{A}C = Q\hat{P}R$

$B\hat{C}A = Q\hat{R}P$.

Example 5.9

Use the following figure to prove that $\Delta ADB \cong \Delta ADC$ and $\overline{DB} = \overline{DC}$.



Proof

Given ΔABC, $\overline{AD}$ is perpendicular to $\overline{BC}$ and it is required to prove that $\Delta ADB \cong \Delta ADC$. From ΔADB and ΔADC it follows that $\overline{AB} = \overline{AC}$ (given)

$A\hat{D}B = A\hat{D}C = 90°$ (given)

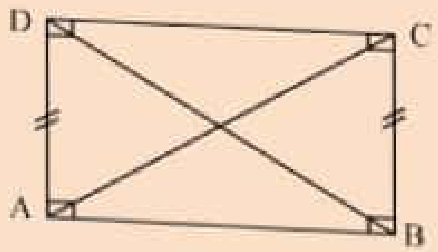
$\overline{AD}$ is a common side to both triangles.

Therefore, $\Delta ADB \cong \Delta ADC$ (by RHS), and $\overline{DB} = \overline{DC}$ (definition of congruence of triangles).

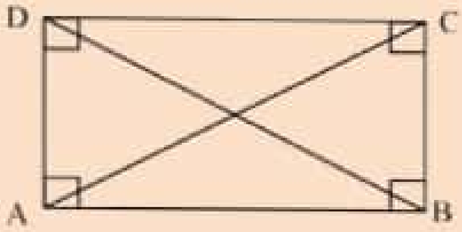
Exercise 5.5

Answer the following questions:

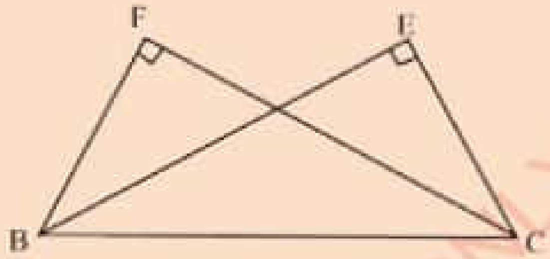
1. In the following figure, $\overline{AC} = \overline{BD}$, prove that $\Delta ABD \cong \Delta BAC$.



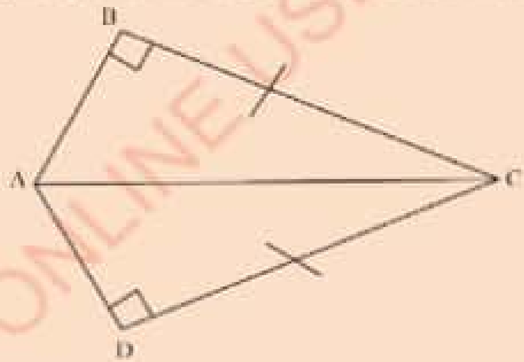
2. Use the following figure to prove that $\overline{AB}$ is parallel to $\overline{DC}$.



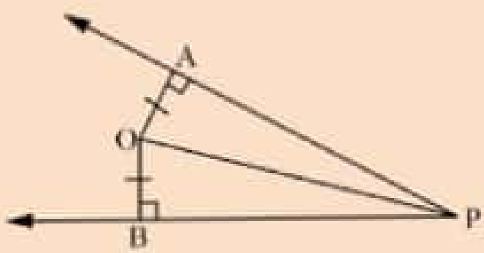
3. In the following figure $\overline{BF} = \overline{CE}$, prove that $\overline{CF} = \overline{BE}$.



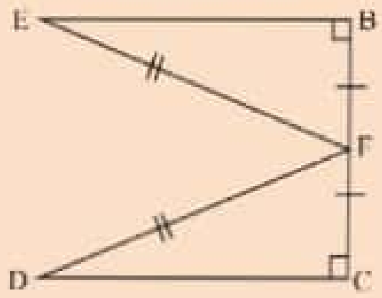
4. In the following figure, prove that $\overline{AC}$ bisects $B\hat{A}D$ and $B\hat{C}D$.



5. Use the following figure to prove that $\overline{PA} = \overline{PB}$.

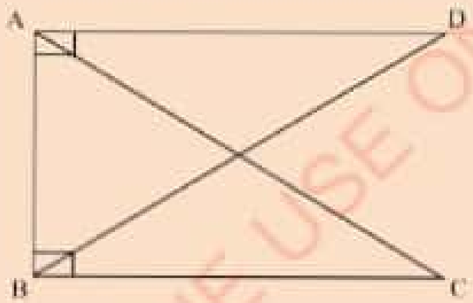


6. Use the following figure to prove that $\overline{BE} = \overline{CD}$.

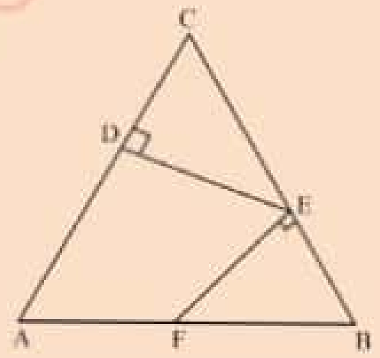


7. If N is the foot of the perpendicular from the centre O of a circle to the chord AB, prove that $\overline{AN} = \overline{NB}$.

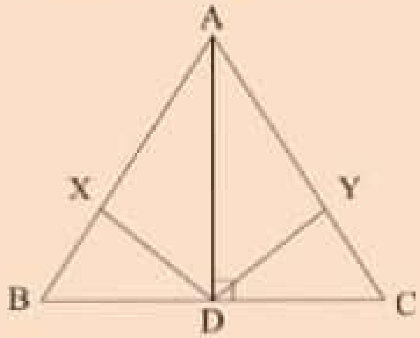
8. Use the following figure to prove that $A\hat{C}B = B\hat{D}A$ if $\overline{AC} = \overline{BD}$



9. In the following figure $\overline{AC} = \overline{BC}$ and $\overline{DE} = \overline{FE}$, prove that $\overline{CD} = \overline{EB}$.



10. In the following figure, D is the midpoint of $\overline{BC}$, X and Y are points on $\overline{AB}$ and $\overline{AC}$ respectively, such that $\overline{DX} = \overline{DY}$ and $D\hat{X}B = D\hat{Y}C = 90°$. Prove that $A\hat{B}D = A\hat{C}D$.



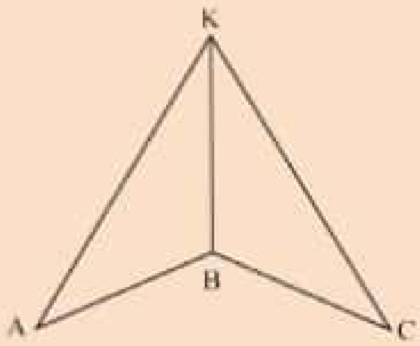
Chapter summary

1. Two figures are said to be congruent if they have exactly the same size and shape.
2. Two triangles are congruent if:
   (a) Three sides of one triangle have equal lengths to the corresponding three sides of the other triangle (SSS).
   (b) The lengths of two sides and the included angle of one triangle are respectively equal to the lengths of two corresponding sides and the included angle of the other triangle (SAS).
   (c) (i) Two angles and the included side of one triangle are respectively equal to the corresponding two angles and the included side of the other triangle (ASA).
   (ii) Two angles and non – included side of one triangle are respectively equal to the corresponding two angles and a non – included side of the other triangle (AAS).
3. Two right – angled triangles are congruent if their hypotenuses and a pair of sides have equal length.
4. To prove that two triangles are congruent use the above congruence postulates.

Revision exercise 5

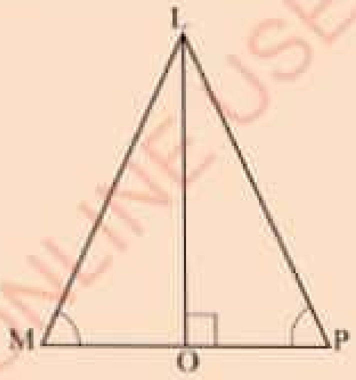
Answer the following questions:

1. In the following figure, $\overline{BA} = \overline{BC}$ and $\overline{KA} = \overline{KC}$. Prove that $B\hat{A}K = B\hat{C}K$.

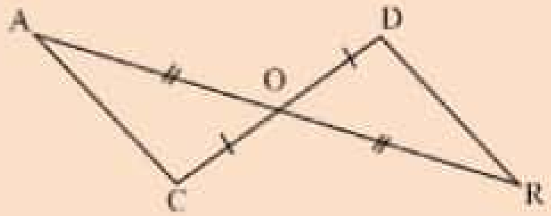


2. A quadrilateral MNOP has the property that $\overline{MN} = \overline{OP}$ and $\overline{MP} = \overline{NO}$. Prove that $P\hat{M}N = N\hat{O}P$.

3. Use the following figure to prove that $\overline{ML} = \overline{PL}$.

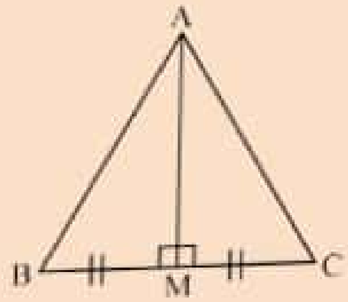


4. In the following figure, prove that $A\hat{C}D = C\hat{D}B$.

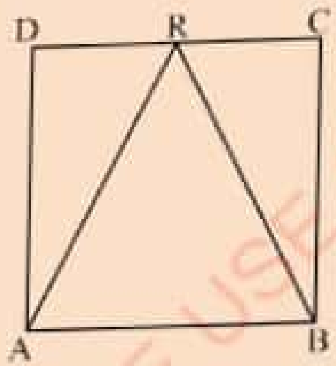


Math F2 – (BOOK) (DUMMY) Sept 4.indb 181

5. Use the following figure to prove that $B\hat{M}A = C\hat{M}A = 90^{\circ}$, given that $\overline{AB} = \overline{AC}$.

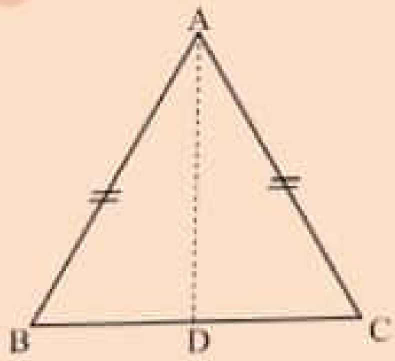


6. If ABCD is a square and $\overline{AR} = \overline{BR}$, prove that R is the midpoint of $\overline{DC}$.



7. Prove that the line segment from the vertical angle of an isosceles triangle to the mid – point of its base is perpendicular to the base.

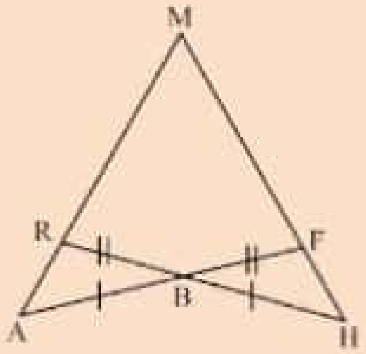
8. In the following figure, prove that $\overline{BD} = \overline{DC}$.



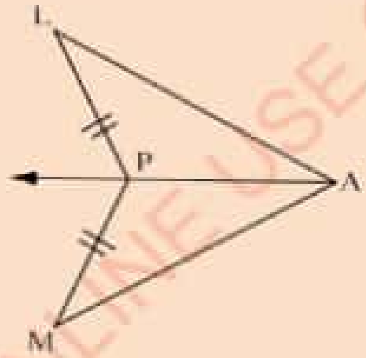
9. Prove that the bisector of the vertical angle of an isosceles triangle is perpendicular to the base at its mid – point.

10. In the following figure $\overline{AB} = \overline{HB}$ and $\overline{RB} = \overline{BF}$. Prove that:

(a) $R\hat{A}B = F\hat{H}B$ (b) $\overline{AM} = \overline{HM}$

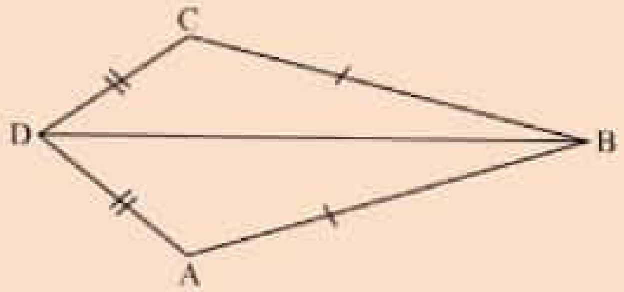


11. Use the following figure to prove that $\overline{AP}$ bisects $L\hat{A}M$, given that $\Delta ALP \cong \Delta AMP$.

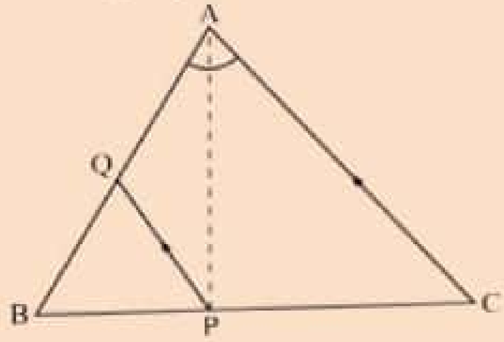


12. Prove that the perpendicular from the vertex to the base of an isosceles triangle bisects the base and the vertical angle.

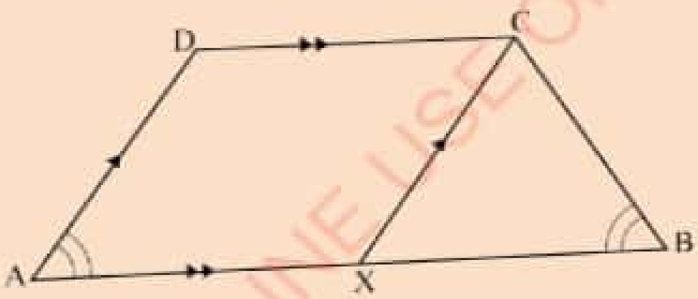
13. Use the following figure to prove that $B\hat{A}D = B\hat{C}D$.



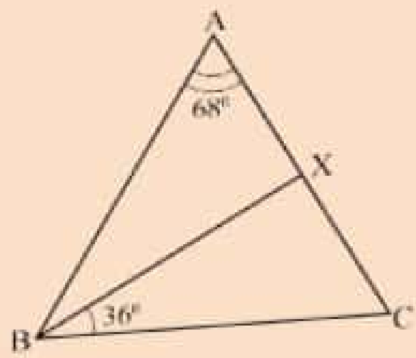
14. If $\overline{PA}$ bisects the angle $B\hat{A}C$ and Q is the point on $\overline{AB}$ such that $\overline{QP} // \overline{AC}$. Prove that $\overline{AQ} = \overline{QP}$.



15. Given the trapezium ABCD such that, $D\hat{A}X = C\hat{B}X$ and $\overline{CX}$ is drawn parallel to $\overline{AD}$ as shown in the following figure. Prove that ABXC is an isosceles triangle.

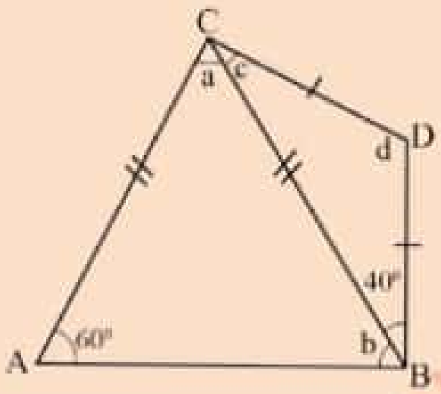


16. If $\overline{AB} = \overline{AC}$, find the value of $A\hat{X}B$ in the following figure.



17. If ΔPQR is an equilateral triangle such that $\overline{PQ}$ is extended to S so that $\overline{QS} = \overline{QR}$. Calculate the value of $Q\hat{R}S$.

18. Use the following figure to find the values of *a*, *b*, *c* and *d*.



Math F2 - (BOOK) (DUMMY) Sept 4.indb 185

Chapter Six

Similarity

Introduction

Geometrical objects or figures are similar if they have the same shape but not necessarily of the same size. A geometrical figure can be obtained from an existing figure by either enlarging, reducing, or by keeping the size unchanged. Knowledge of similarity is the basis of all measurements as it deals with map – making, scale drawings and also explains some aspects of photographic images. In this chapter, you will learn about similar figures including triangles, proofs of similarity theorems and properties of similar triangles. The competences developed in this chapter will enable you to apply the concept of similarity in architectural matters such as finding heights of buildings, bridges and trees where tape measures cannot be used conveniently.

Similar figures

Activity 6.1: Recognizing similarities between figures

Materials required: Manila paper, ruler, pencil

1. Draw three rectangles of different sizes.
   (a) What is the length and width of each figure?
   (b) Are the figures of the same shape? Explain.
   (c) Is there any relationship between the lengths of the sides of the rectangles you have drawn?
2. Take a piece of paper and make two triangles by splitting them in such a way that, each side of the second triangle is twice the length of the corresponding side of the first triangle. What can you say about the two triangles?

3. Is there any similarities between the following figures? Why?

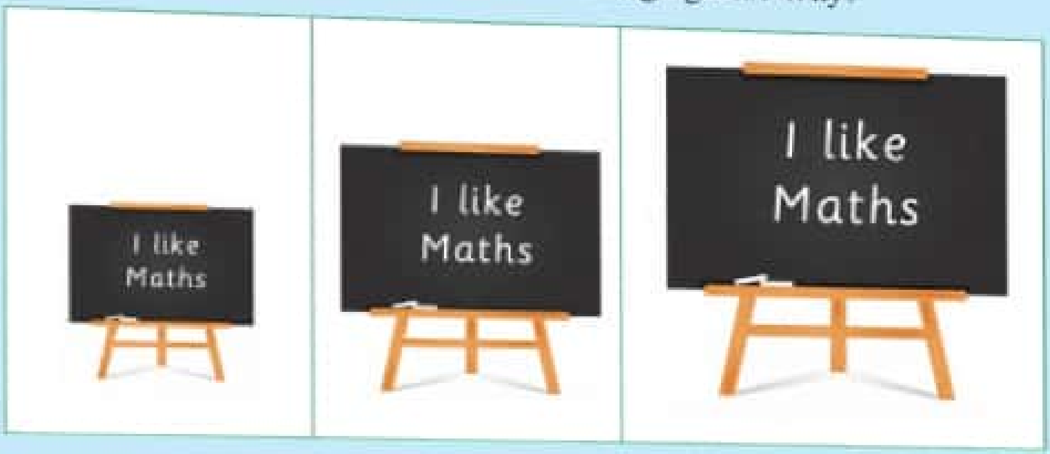


Figure 6.1: *Objects with the same shape*

Two geometrical objects are said to be similar if they both have the same shape or one has the same shape as the smaller image of the other. More precisely, one of the objects can be obtained from the other by uniform scaling (enlarging or shrinking). That is, either object can be scaled, repositioned or reflected to coincide precisely with the other object. Two polygons are similar if their corresponding angles are equal (equiangular) and corresponding sides are proportional. Figure 6.2, shows two polygons PQRS and TUVW that are similar.

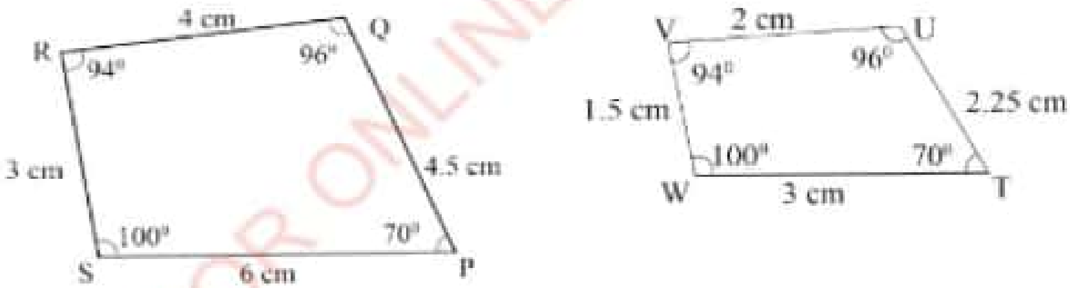


Figure 6.2: *Similar polygons*

Polygon PQRS shows that $S\hat{P}Q$ corresponds to $W\hat{T}U$ in polygon TUVW and each measures 70°, $P\hat{Q}R$ corresponds to $T\hat{U}V$ in TUVW and each measures 96°, $Q\hat{R}S$ corresponds to $U\hat{V}W$ and each measures 94°, and $R\hat{S}P$ corresponds to $V\hat{W}T$ and each measures 100°. Thus, the polygons satisfy the condition that corresponding angles are equal. Also $\overline{PQ}$ in PQRS corresponds to $\overline{TU}$ in TUVW,

$\overline{QR}$ corresponds to $\overline{UV}$, $\overline{RS}$ corresponds to $\overline{VW}$, and $\overline{SP}$ corresponds to $\overline{WT}$. Each pair of these corresponding sides have the same ratio. That is,

$$\frac{\overline{PQ}}{\overline{TU}} = \frac{4.5 \text{ cm}}{2.25 \text{ cm}} = 2, \ \frac{\overline{QR}}{\overline{UV}} = \frac{4 \text{ cm}}{2 \text{ cm}} = 2, \ \frac{\overline{RS}}{\overline{VW}} = \frac{3 \text{ cm}}{1.5 \text{ cm}} = 2 \text{ and } \frac{\overline{SP}}{\overline{WT}} = \frac{6 \text{ cm}}{3 \text{ cm}} = 2.$$

The value of the constant ratio is called the **constant of proportionality** or **scale factor.** The constant of proportionality indicates that corresponding sides are proportional, hence the similarity of the two polygons. In general, plane figures are similar when they are equiangular although they may differ in size. In geometry, a polygon which has all of its sides equal and all of its angles equal is called a **regular polygon.** Therefore, all regular polygons with a given number of sides are similar.

Similar triangles

Triangles are similar when their corresponding angles are equal and their corresponding sides are proportional. Consider the pair of triangles shown in Figure 6.3.

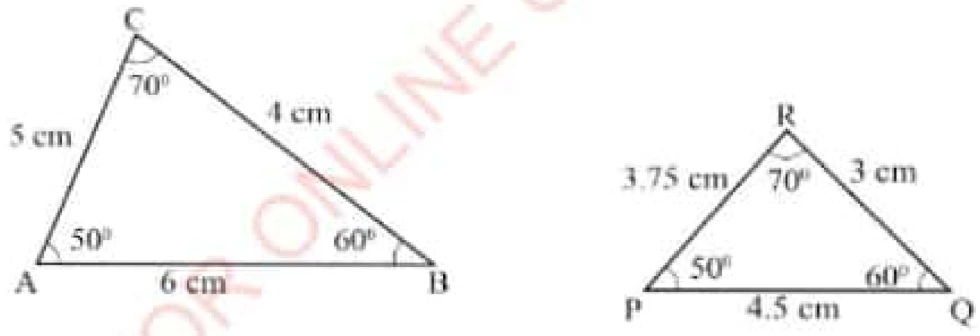


Figure 6.3: *Similar triangles*

$C\hat{A}B$ of ΔABC corresponds to $R\hat{P}Q$ of ΔPQR and each measures 50°, $A\hat{B}C$ corresponds to $P\hat{Q}R$ and each measures 60°, and $B\hat{C}A$ corresponds to $Q\hat{R}P$ and each measures 70°.

Since the corresponding angles are equal, then the two triangles are similar. Also, $\overline{AB}$ corresponds to $\overline{PQ}$, $\overline{BC}$ corresponds to $\overline{QR}$, and $\overline{CA}$ corresponds to $\overline{RP}$. The ratios of the corresponding sides are given by:

$$\frac{\overline{AB}}{\overline{PQ}} = \frac{6\text{ cm}}{4.5\text{ cm}} = \frac{4}{3},\quad \frac{\overline{BC}}{\overline{QR}} = \frac{4\text{ cm}}{3\text{ cm}} = \frac{4}{3} \text{ and } \frac{\overline{CA}}{\overline{RP}} = \frac{5\text{ cm}}{3.75\text{ cm}} = \frac{4}{3}.$$

Therefore, $\frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{BC}}{\overline{QR}} = \frac{\overline{CA}}{\overline{RP}} = \frac{4}{3}$.

Since the corresponding sides are also proportional, then the two triangles are similar. Therefore, ΔABC is similar to ΔPQR, denoted by $\Delta ABC \sim \Delta PQR$. The symbol $\sim$ means **similar to**. Similar triangles (polygons) are named corresponding to the order of their vertices.

For example, if ΔGHK is similar to ΔXYZ, then it can be deduced from the order of the vertices that $\overline{GH}$ of the first triangle corresponds to $\overline{XY}$ of the second triangle. Therefore, $\overline{GK}$ corresponds to $\overline{XZ}$ and $\overline{HK}$ corresponds to $\overline{YZ}$.

Example 6.1

Given that $\Delta SLK \sim \Delta NFR$ identify all the corresponding angles and the corresponding sides.

Solution

Using the order of vertices used to name the two similar triangles, $S\hat{L}K$ corresponds to $N\hat{F}R$, $K\hat{S}L$ corresponds to $R\hat{N}F$, and $L\hat{K}S$ corresponds to $F\hat{R}N$. Also, $\overline{LS}$ of ΔSLK corresponds to $\overline{FN}$ of ΔNFR, $\overline{SK}$ of ΔSLK corresponds to $\overline{NR}$ of ΔNFR, and $\overline{KL}$ of ΔSLK corresponds to $\overline{RF}$ of ΔNFR.

Example 6.2

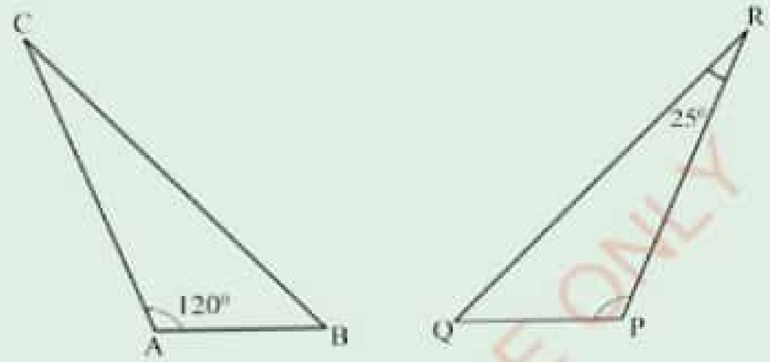
Given that $\Delta ABC \sim \Delta PQR$, find the value of $A\hat{B}C$ if;

(a) $B\hat{A}C = 120°$ and $P\hat{R}Q = 25°$

(b) $Q\hat{P}R + B\hat{C}A = 145°$

Solution

Consider the following figures of ΔABC and ΔPQR.



(a) Since $Q\hat{R}P$ corresponds to $B\hat{C}A$, then $Q\hat{R}P = B\hat{C}A = 25°$.

In ΔABC, $A\hat{B}C + B\hat{C}A + B\hat{A}C = 180°$ (sum of interior angles in a triangle)

$A\hat{B}C + 25° + 120° = 180°$

Therefore, $A\hat{B}C = 35°$.

(b) Since $Q\hat{P}R$ corresponds to $C\hat{A}B$, then $Q\hat{P}R = C\hat{A}B$.

Thus, $Q\hat{P}R + B\hat{C}A = C\hat{A}B + B\hat{C}A = 120° + 25° = 145°$

But, $A\hat{B}C + B\hat{A}C + A\hat{C}B = 180°$ (sum of interior angles in a triangle).

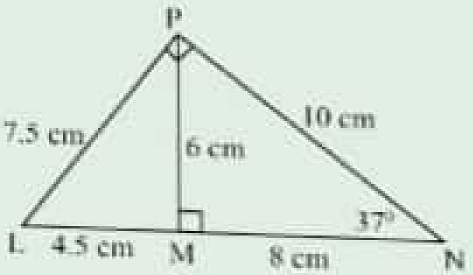
Then, $145° + A\hat{B}C = 180°$

$A\hat{B}C = 180° - 145° = 35°$.

Therefore, $A\hat{B}C = 35°$.

Example 6.3

In the following figure, name the triangles which are similar and determine the constant of proportionality needed to show their similarity.



Solution

Consider ΔPMN and ΔLPN

$M\hat{N}P = L\hat{N}P$ (common)

$N\hat{M}P = L\hat{P}N$ (each 90°, given).

$M\hat{P}N = P\hat{L}N$ (third angles of the triangles)

Therefore, $\Delta PMN \sim \Delta LPN$ (1)

Consider ΔLMP and ΔLPN

$M\hat{L}P = P\hat{L}N$ (common)

$L\hat{M}P = N\hat{P}L$ (each 90°, given)

$L\hat{P}M = L\hat{N}P$ (third angles of the triangles)

$\Delta LMP \sim \Delta LNP$ (2)

Relating (1) and (2) we find that

$\Delta LPN \sim \Delta PMN \sim \Delta LMP$

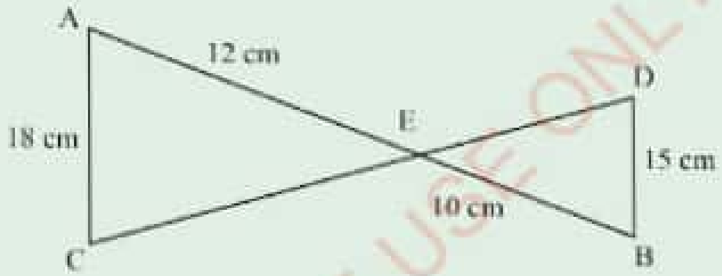
Since, $\Delta LPN \sim \Delta PMN$, then the constant of proportionality is given by

$$\frac{\overline{LP}}{\overline{PM}}=\frac{\overline{PN}}{\overline{MN}}=\frac{\overline{NL}}{\overline{NP}}. \quad \text{Thus, } \frac{7.5\text{ cm}}{6\text{ cm}}=\frac{10\text{ cm}}{8\text{ cm}}=\frac{12.5\text{ cm}}{10\text{ cm}}=\frac{5}{4}.$$

Therefore, the constant of proportionality ratio needed to show their similarity is $5:4$ or $\frac{5}{4}$.

Example 6.4

In the following figure, find the constant of proportionality needed to obtain a pair of similar triangles, if $\overline{CE}:\overline{DE} = 1.2:1$. Name this pair of similar triangles.



Solution

The ratio of lengths of corresponding sides are given by;

$$\overline{CE}:\overline{DE}=\frac{\overline{CE}}{\overline{DE}}=\frac{1.2}{1}=\frac{12\text{ cm}}{10\text{ cm}}=\frac{6}{5}$$

$$\overline{AE}:\overline{BE}=\frac{\overline{AE}}{\overline{BE}}=\frac{12\text{ cm}}{10\text{ cm}}=\frac{6}{5}$$

$$\overline{AC}:\overline{BD}=\frac{\overline{AC}}{\overline{BD}}=\frac{18\text{ cm}}{15\text{ cm}}=\frac{6}{5}$$

$$\text{Thus, } \frac{\overline{CE}}{\overline{DE}}=\frac{\overline{AE}}{\overline{BE}}=\frac{\overline{AC}}{\overline{BD}}=\frac{6}{5}.$$

Therefore, $\Delta ACE \sim \Delta BDE$ (corresponding sides are proportional) and $\frac{6}{5}$ is the constant of proportionality.

Exercise 6.1

Answer the following questions:

1. (a) Given that $\Delta PQR \sim \Delta TSM$, identify the corresponding angles and the corresponding sides.

   (b) Given that $\Delta PQR \sim \Delta LMN$ and $\Delta PQR \sim \Delta ABC$, identify the corresponding angles and corresponding sides between ΔABC and ΔLMN.

2. (a) One rectangle has length 10 cm and width 5 cm. Another rectangle has length 13 cm and width 4 cm. Determine whether or not the two rectangles are similar. Explain your answer.

   (b) A rectangle has length16 cm and width 23 cm. A second rectangle has length 12 cm and width 9 cm. Are the two rectangles similar? Explain your answer.

3. Given that ΔABC and ΔLMN are similar, find the value of $A\hat{C}B$, when

   (a) $A\hat{B}C = 70°$ and $M\hat{N}L = 40°$

   (b) $A\hat{B}C + M\hat{L}N = 130°$

   (c) $L\hat{N}M$ and $B\hat{A}C$ are complementary

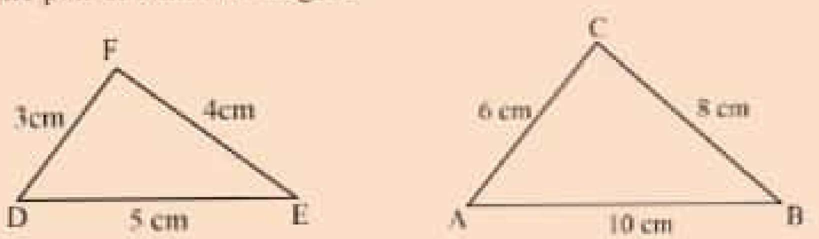
4. Given that $\frac{\overline{AB}}{\overline{KL}} = 2$, $\frac{\overline{BT}}{\overline{LS}} = 2$ and $\frac{\overline{TA}}{\overline{SK}} = 2$,

   (a) name the triangles which are similar.

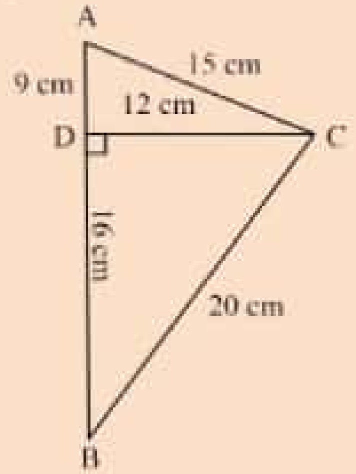
   (b) identify the corresponding angles.

5. In each pair of the following figures, determine the constant of proportionality so that the pair of the triangles are similar. In each case, state the pair of similar triangles.

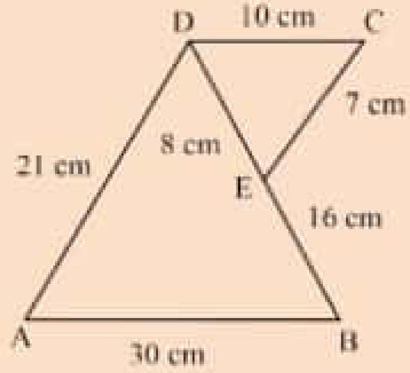
   (a)



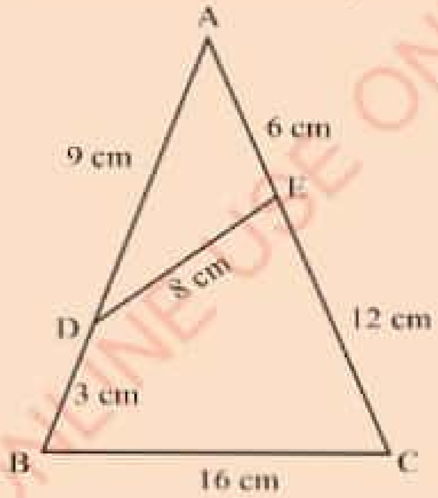
(b)



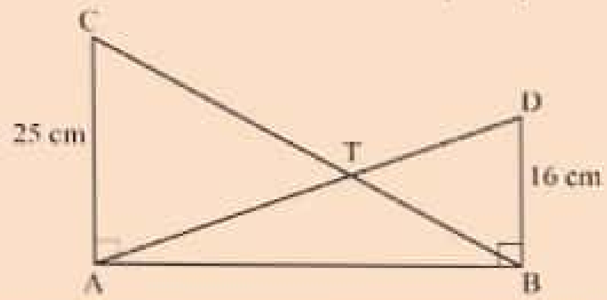
(c)



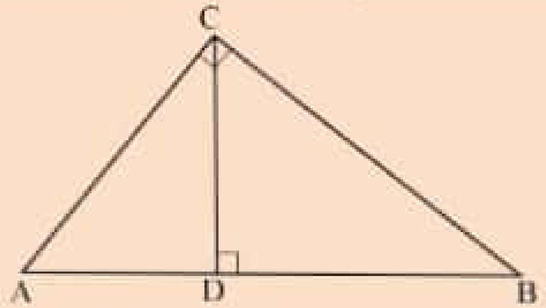
6. Use the vertices A, D, B, C and E of the following figure to identify all similar triangles, and hence state their corresponding angles.



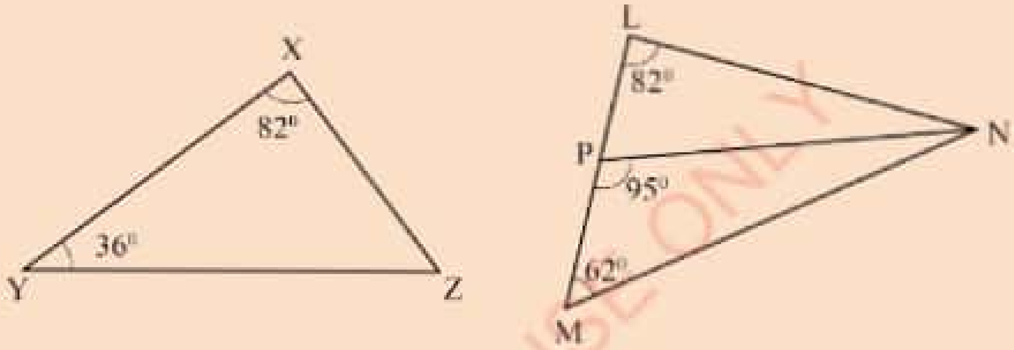
7. Does the following figure have sufficient information to determine whether ΔACT and ΔDBT are similar? Explain your answer.



8. In the following figure, name the triangles which are similar to ΔADC.



9. In figure LMN, name the triangle which is similar to ΔXYZ.



10. Which of the following figures are always similar?

(a) Circles (b) Hexagons

(c) Rhombuses (d) Rectangles

(e) Squares (f) Congruent polygons

Angle – Angle (AA) similarity theorem

The AA – similarity theorem states that, if a correspondence between two triangles is such that two pairs of corresponding angles are equal, then the two triangles are similar. Figure 6.3 shows two similar triangles by the AA – similarity theorem.

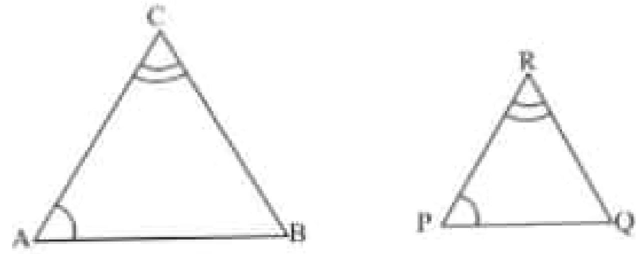


Figure 6.3: *Similar triangle by the Angle – Angle theorem*

In Figure 6.3, we observe that, $B\hat{A}C = Q\hat{P}R$, $A\hat{C}B = P\hat{R}Q$

Thus, $A\hat{B}C = P\hat{Q}R$ (third angles of triangles)

Therefore, $\Delta ABC \sim \Delta PQR$ and $\frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{BC}}{\overline{QR}} = \frac{\overline{AC}}{\overline{PR}}$.

Side – Side – Side (SSS) - similarity theorem

The SSS – similarity theorem states that, if the correspondence between two triangles is such that the corresponding lengths of the sides are proportional, then the triangles are similar. Figure 6.4 shows two similar triangles by the SSS – similarity theorem.

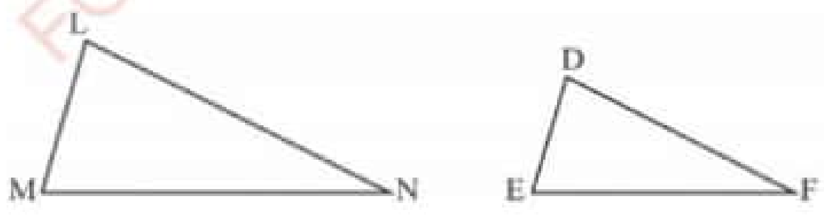


Figure 6.4: *Similar triangle by the Side – Side – Side theorem*

In Figure 6.4, we observe that,

$\frac{\overline{LM}}{\overline{DE}} = \frac{\overline{LN}}{\overline{DF}} = \frac{\overline{MN}}{\overline{EF}}$, then $\Delta LMN \sim \Delta DEF$

Therefore, $L\hat{M}N = D\hat{E}F$, $M\hat{N}L = E\hat{F}D$ and $M\hat{L}N = E\hat{D}F$.

Side – Angle – Side (SAS) similarity theorem

The SAS – similarity theorem states that, if a correspondence between two triangles is such that two pairs of corresponding sides are proportional and the included angles are equal, then the triangles are similar. Figure 6.5 shows two similar triangles by the SAS – similarity theorem.

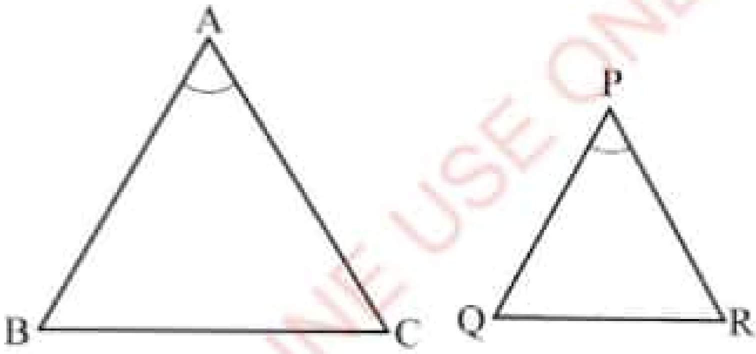


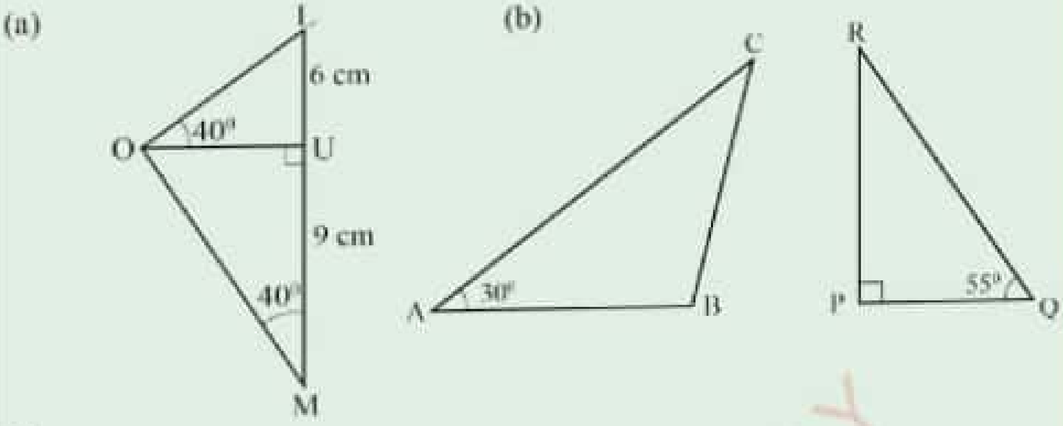
Figure 6.5: *Similar triangles by Side – Angle – Side theorem*

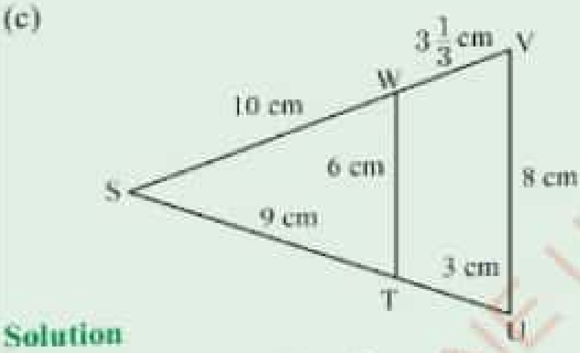
In Figure 6.5, if $\frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{AC}}{\overline{PR}}$ and $B\hat{A}C = Q\hat{P}R$, then $\Delta ABC \sim \Delta PQR$.

Therefore, $A\hat{B}C = P\hat{Q}R$ and $A\hat{C}B = P\hat{R}Q$.

Example 6.5

For each pair of triangles in the following figures, determine whether they are similar or not. Indicate the similarity theorem used to support your argument.





Solution

In figure (a), observation shows that,

$L\hat{O}U = O\hat{M}U = 40°$ (given)

$O\hat{U}L = M\hat{U}O = 90°$ (given)

$U\hat{L}O = U\hat{O}M = 50°$ (third angles in triangles)

Therefore, $\Delta OUL \sim \Delta MUO$ (by SAS – similarity theorem)

In figure (b), the triangles are not similar because the corresponding angles are not equal.

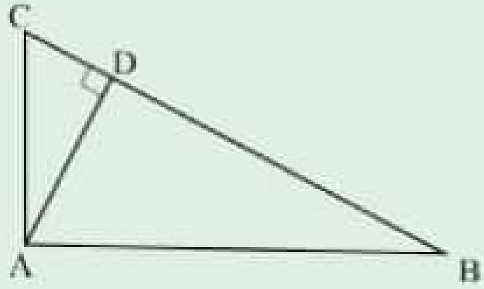
In figure (c), observation shows that,

$$\frac{\overline{SW}}{\overline{SV}} = \frac{10 \text{ cm}}{13\frac{1}{3} \text{ cm}} = \frac{3}{4}, \frac{\overline{ST}}{\overline{SU}} = \frac{9 \text{ cm}}{12 \text{ cm}} = \frac{3}{4}, \text{ and } \frac{\overline{TW}}{\overline{UV}} = \frac{6 \text{ cm}}{8 \text{ cm}} = \frac{3}{4}$$

Therefore, $\Delta SVU \sim \Delta SWT$ (by SSS – Similarity theorem).

Example 6.6

Study the following figure and answer the questions that follows.



Use the figure to prove that,

(a) $\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{CD}}{\overline{AC}}$

(b) $\frac{\overline{DB}}{\overline{DA}} = \frac{\overline{BA}}{\overline{AC}}$

Solution

(a) The numerators contain the vertices A, D and C while the denominators contain A, B and C. Thus, the triangles ADC and ABC are obtained.

In ΔADC and ΔABC we have,

$A\hat{D}C = C\hat{A}B$ (each measures $90°$)

$A\hat{C}D = A\hat{C}B$ (common)

$C\hat{A}D = A\hat{B}C$ (third angles of triangles)

Thus, $\Delta DCA \sim \Delta ACB$ (by AA - similarity theorem).

The ratios of the corresponding sides are given by:

$$\frac{\overline{CD}}{\overline{AC}} = \frac{\overline{CA}}{\overline{CB}} = \frac{\overline{DA}}{\overline{AB}}$$

Therefore, $\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{DC}}{\overline{AC}}$ or $\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{CD}}{\overline{AC}}$.

(b) Vertices appearing in the numerators are A, B and D, while in the denominators are A, D and C. Thus, the triangles ADC and BDA are obtained.

In ΔADC and ΔBDA, we have

$A\hat{D}B = A\hat{D}C$ (each measures 90°)

$A\hat{B}D = C\hat{A}D = A\hat{B}C$ (using the proof in (a) above)

$D\hat{A}B = A\hat{C}D$ (third angles of triangles)

Thus, $\Delta BDA \sim \Delta ADC$ (by AA - similarity theorem)

The ratios of the corresponding sides are given by:

$$\frac{\overline{DB}}{\overline{DA}} = \frac{\overline{BA}}{\overline{AC}} = \frac{\overline{AD}}{\overline{CD}}$$

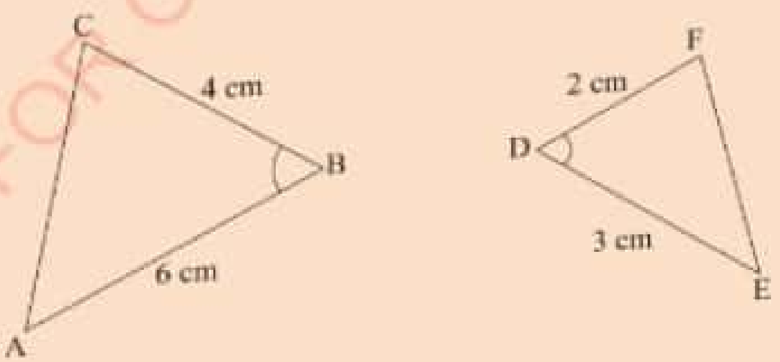
Therefore, $\frac{\overline{DB}}{\overline{DA}} = \frac{\overline{BA}}{\overline{AC}}$.

Exercise 6.2

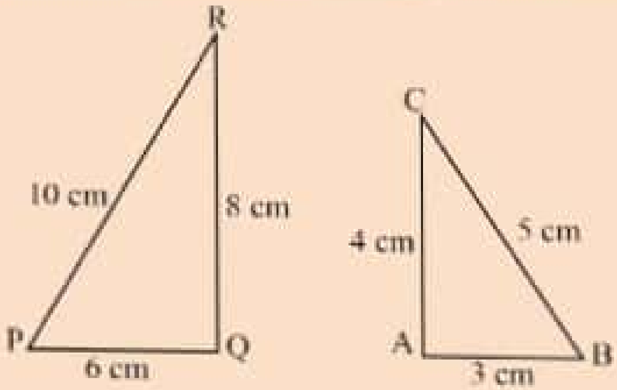
Answer the following questions:

1. In each pair of the following figures, determine whether they are similar or not. Indicate the similarity theorem used to support your argument.

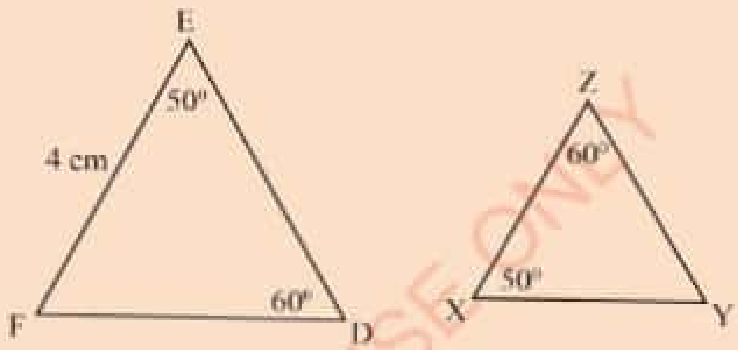
(a)



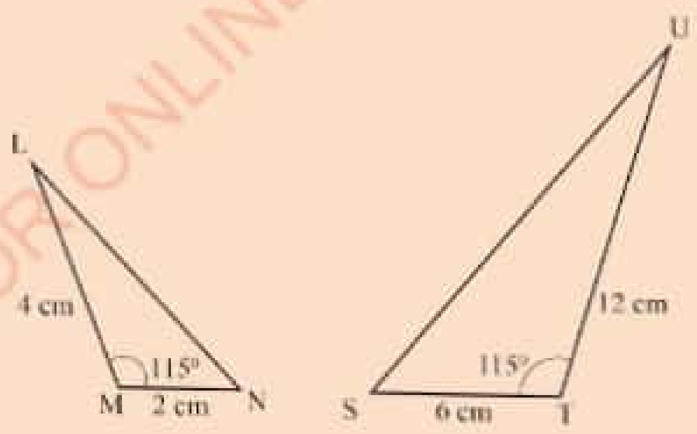
(b)


R
10 cm
8 cm
P
6 cm
Q
C
4 cm
5 cm
A
3 cm
B

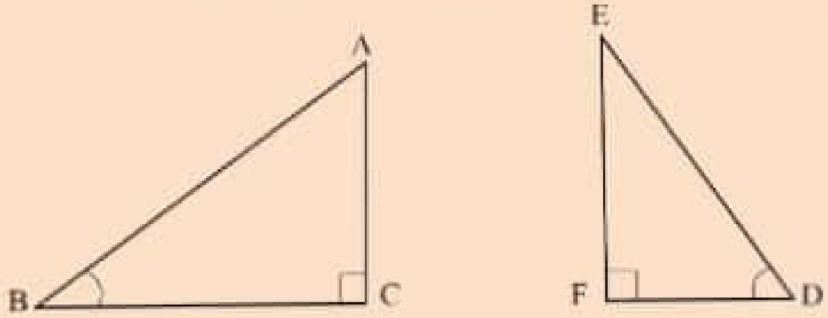
(c)


E
50°
4 cm
F
60°
D
Z
60°
50°
X
Y

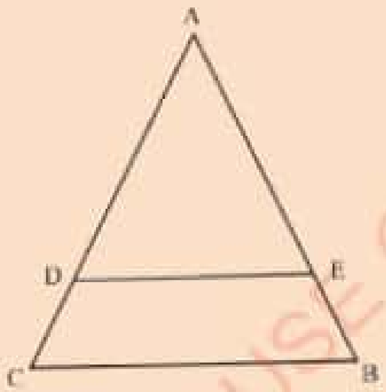
(d)


L
4 cm
115°
M
2 cm
N
U
12 cm
115°
S
6 cm
T

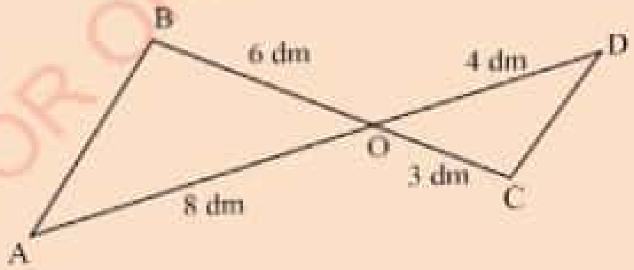
(e)



2. In the following figure, $\frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$. Prove that $\Delta ABC \sim \Delta AED$.

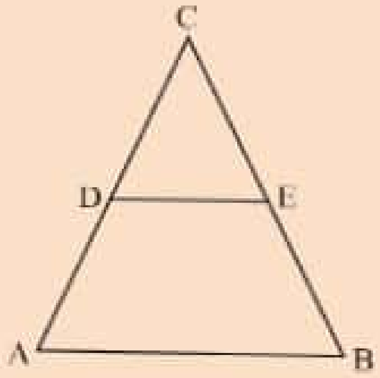


3. In the following figure, if $\overline{OA} = 8$ dm, $\overline{BO} = 6$ dm, $\overline{OD} = 4$ dm and $\overline{OC} = 3$ dm. Prove that $\Delta AOB \sim \Delta DOC$.

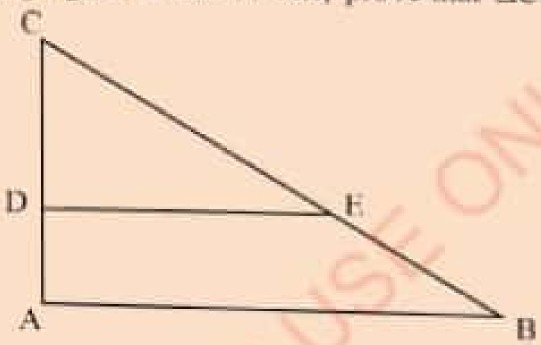


4. The lengths of the legs of one right – angled triangle are 3 dm and 4 dm. The lengths of the legs of another right – angled triangle are 5 cm and 8 cm. Are the two triangles similar? Explain.

5. In the following figure, if $\Delta CDE \sim \Delta CAB$, prove that $\overline{DE} \parallel \overline{AB}$.

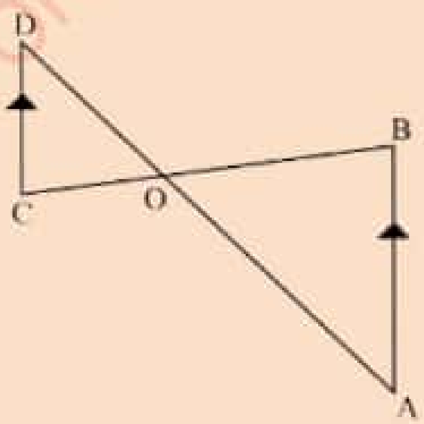


6. In the following figure, if $\overline{DE} \parallel \overline{AB}$, prove that $\Delta CDE \sim \Delta CBA$.

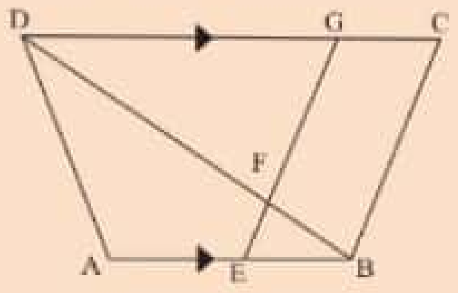


7. In the following figure, if $\overline{CD} \parallel \overline{AB}$, then prove that:

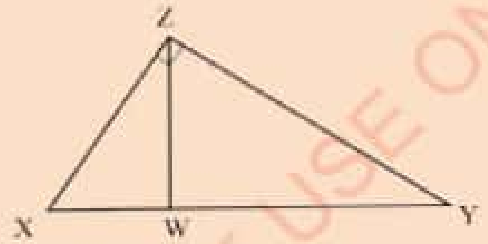
(a) $\Delta ABO \sim \Delta DCO$ (b) $\frac{\overline{AO}}{\overline{OD}} = \frac{\overline{BO}}{\overline{OC}}$



8. In the following figure, E is a point on $\overline{AB}$ and G is a point on $\overline{DC}$ and $\overline{AB}$ // $\overline{DC}$. Prove that $\frac{\overline{DG}}{\overline{BE}} = \frac{\overline{FD}}{\overline{FB}}$.

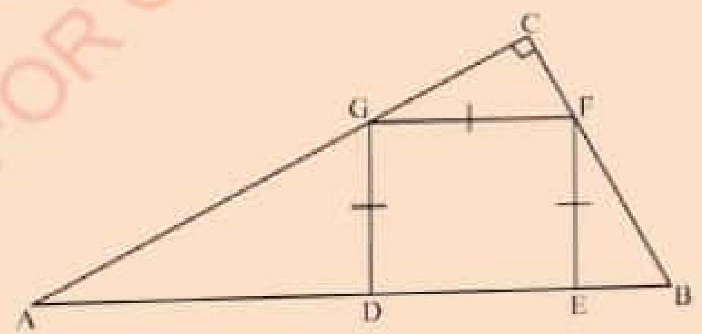


9. In the following figure, ΔXYZ is a right – angled triangle and $\frac{\overline{XW}}{\overline{XZ}} = \frac{\overline{XZ}}{\overline{XY}}$. Prove that $\overline{ZW}$ is perpendicular to $\overline{XY}$.

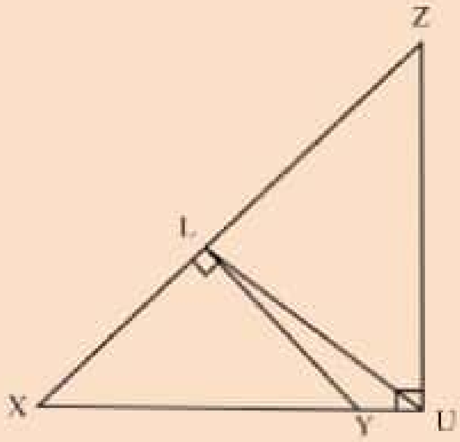


10. In the following figure, DEFG is a square and $A\hat{C}B$ is a right angle. Prove that:

(a) $\Delta CFG \sim \Delta DGA$ (b) $\Delta EBF \sim \Delta CFG$

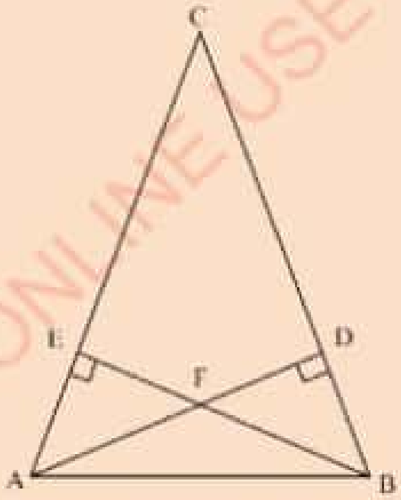


11. Name two similar triangles in the following figure.



12. Use the following figure to prove that:

(a) $\Delta ADC \sim \Delta BEC$ (b) $\frac{\overline{AF}}{\overline{BF}} = \frac{\overline{EA}}{\overline{DB}}$



Properties of similar triangles

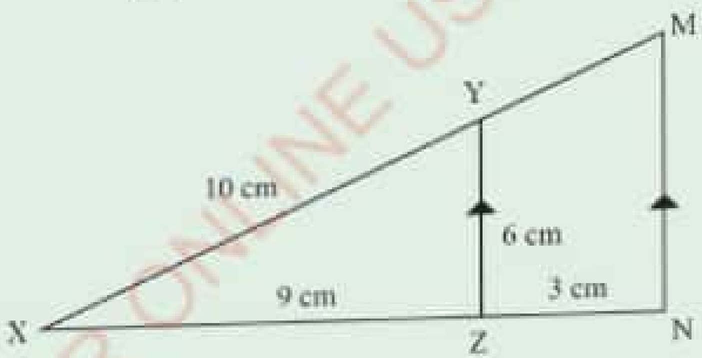
From the previous discussion, properties of similar triangles can be summarised as follows:

1. Corresponding angles of similar triangles are equal.
2. Corresponding lengths of sides of similar triangles are proportional.
3. Two triangles are similar if two angles of one triangle are respectively equal to two corresponding angles of the other.
4. Two triangles are similar if an angle of one triangle is equal to an angle of the other triangle, and the lengths of sides including these angles are proportional.

Example 6.7

In the following figure, calculate:

(a) $\overline{MY}$ (b) $\overline{MN}$



Solution

(a) Since $\overline{ZY}//\overline{MN}$, then we have,

$X\hat{Y}Z = X\hat{M}N$ (corresponding angles)

$X\hat{Z}Y = X\hat{N}M$ (corresponding angles)

Hence, $\Delta XYZ \sim \Delta XMN$ (by AA - similarity theorem)

Thus, $\frac{\overline{XY}}{\overline{XM}}=\frac{\overline{XZ}}{\overline{XN}}=\frac{\overline{YZ}}{\overline{MN}}$, this means that,

$$\frac{10\text{ cm}}{\overline{XM}}=\frac{9\text{ cm}}{12\text{cm}}=\frac{6\text{ cm}}{\overline{MN}}$$

From $\frac{10\text{ cm}}{\overline{XM}}=\frac{9\text{ cm}}{12\text{cm}}$, $\overline{XM}=\frac{10\text{ cm}\times 12\text{ cm}}{9\text{ cm}}=13\frac{1}{3}$ cm.

But $\overline{MY}=\overline{XM}-\overline{XY}=13\frac{1}{3}\text{ cm}-10\text{ cm}=3\frac{1}{3}$ cm

Therefore, $\overline{MY}=3\frac{1}{3}$ cm.

(b) From $\frac{6\text{ cm}}{\overline{MN}}=\frac{9\text{ cm}}{12\text{cm}}$, $\overline{MN}=\frac{6\text{ cm}\times 12\text{ cm}}{9\text{ cm}}=8$ cm.

Therefore, $\overline{MN}=8$ cm.

Example 6.8

In the following figure, $\Delta ABC \sim \Delta PQR$. If $\overline{AC}$ = 4.8 cm, $\overline{AB}$ = 4 cm and $\overline{PQ}$ = 9 cm, find the value of $\overline{PR}$.

Solution

Given that $\Delta ABC \sim \Delta PQR$, then we have,

$$\frac{\overline{AB}}{\overline{PQ}}=\frac{\overline{AC}}{\overline{PR}}$$

That is, $\frac{4\text{ cm}}{9\text{ cm}} = \frac{4.8\text{ cm}}{\overline{PR}} \Rightarrow 4\text{cm} \times \overline{PR} = 9\text{ cm} \times 4.8\text{ cm}$

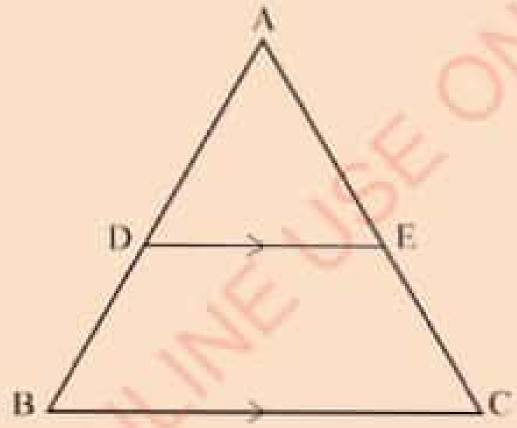
$$\overline{PR} = \frac{9\text{ cm} \times 4.8\text{ cm}}{4\text{cm}} = 10.8\text{ cm}.$$

Therefore, $\overline{PR} = 10.8$ cm.

Exercise 6.3

Answer the following questions:

1. Study the following figure where $\overline{DE}$ is parallel to $\overline{BC}$. Answer the following questions.



(a) If $\overline{AD} = 4$ dm, $\overline{AB} = 8$ dm and $\overline{DE} = 10$ dm, find the value of $\overline{BC}$.

(b) If $\overline{AE} = 5$ cm, $\overline{AC} = 15$ cm, $\overline{BC} = 24$ cm, find the value of $\overline{DE}$.

(c) If $\overline{AD} = 7$ dm, $\overline{DE} = 11$ dm, $\overline{BC} = 22$ dm, find the value of $\overline{BD}$.

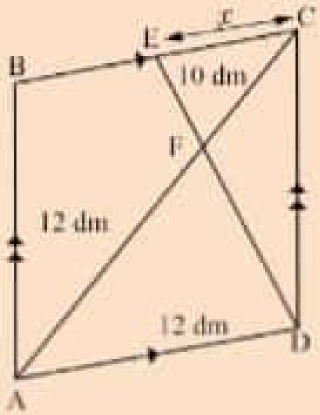
(d) If $\overline{BD} = 9$ dm, $\overline{DE} = 20$ dm, $\overline{BC} = 35$ dm, find the value of $\overline{AD}$.

(e) If $\overline{AD} = 10$ cm, $\overline{DE} = 24$ cm, $\overline{BC} = 84$ cm, find the value of $\overline{AB}$.

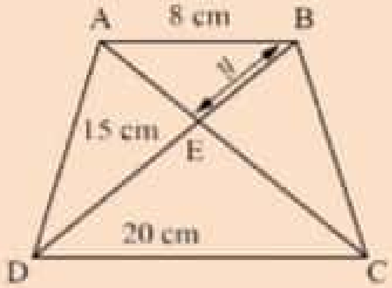
(f) If $\overline{AE} = 3$ cm, $\overline{DE} = 3$ cm, $\overline{BC} = 7$ cm, find the value of $\overline{CE}$.

2. Determine the values of x and y in the following figures:

(a)

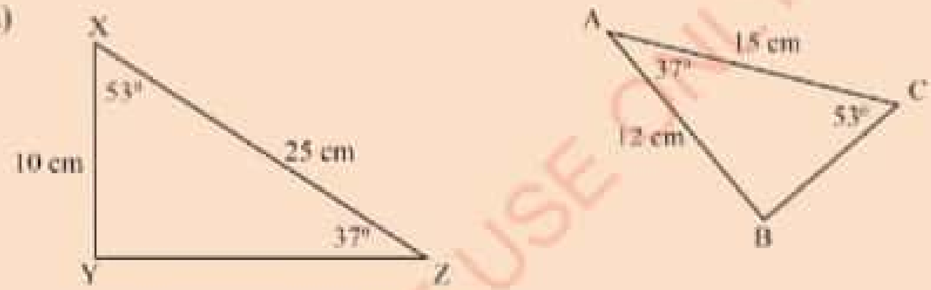


(b)

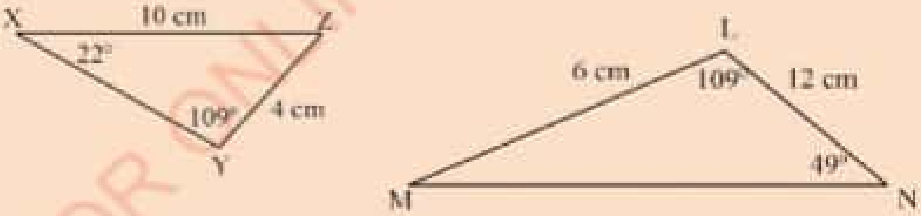


3. In each pair of the following figures, mention the triangle similar to ΔXYZ. Calculate the lengths of the sides and angles which are not given.

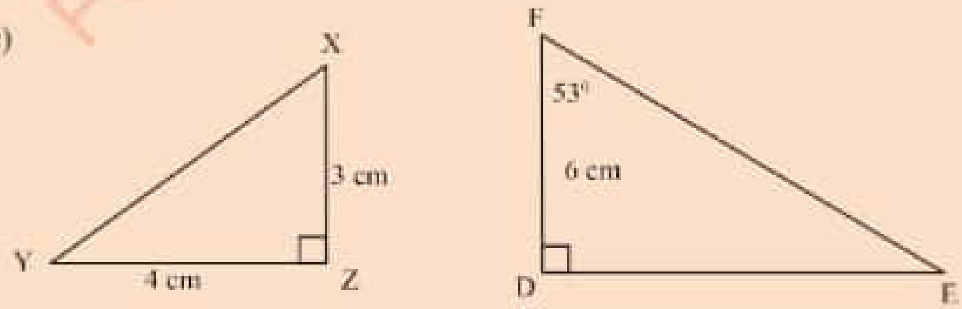
(a)



(b)

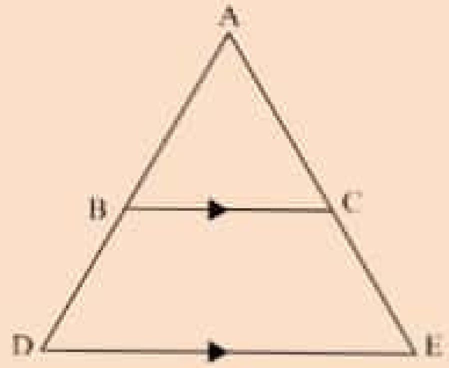


(c)

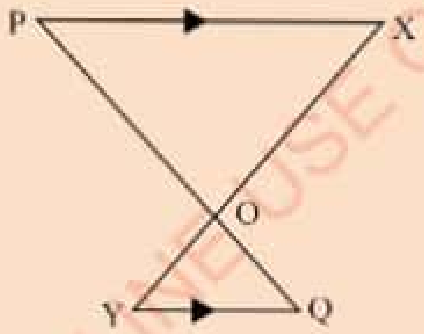


4. In the following figure, state why ΔABC and ΔADE are similar?

   If $\overline{AB} = 8$ dm, $\overline{AC} = 9$ dm, $\overline{BC} = 6$ dm and $\overline{AD} = 12$ dm, calculate the values of $\overline{AE}$ and $\overline{DE}$.



5. Study the following figure and answer the questions that follow.



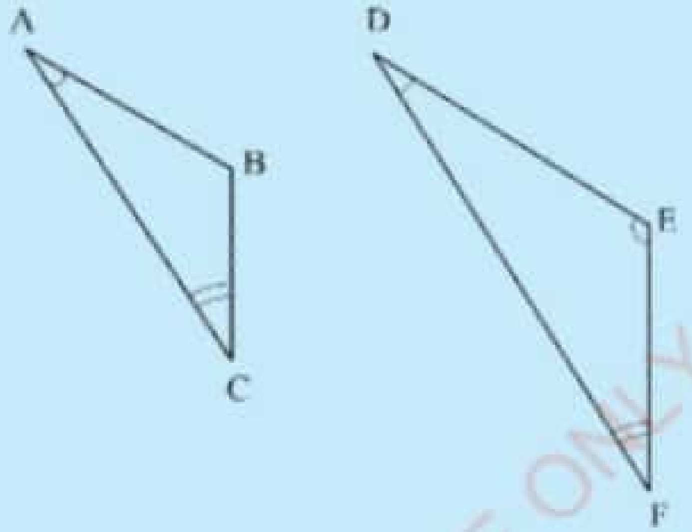
   (a) Which triangle is similar to ΔYOQ?

   (b) Given that $\overline{OP} = 4$ m, $\overline{OX} = 7$ m, $\overline{PX} = 6$ m, $\overline{YQ} = 4.5$ m, calculate the value of $\overline{OY}$ and $\overline{OQ}$.

Chapter summary

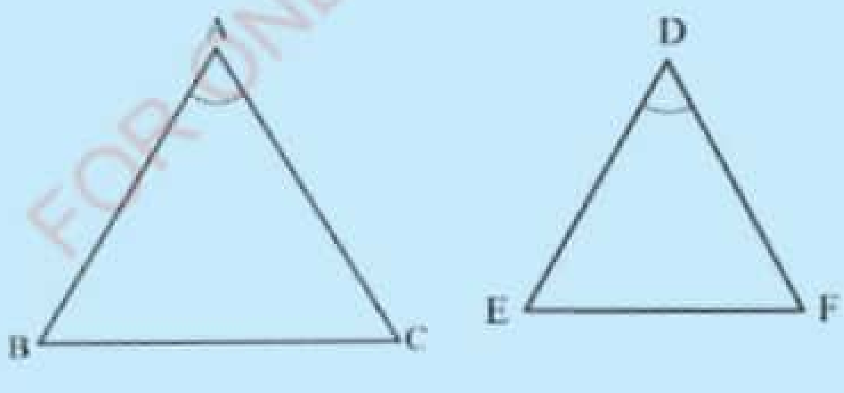
1. AA Similarity Theorem

If the correspondence between triangles is such that two pairs of corresponding angles are congruent, then $\Delta ABC \sim \Delta DEF$.



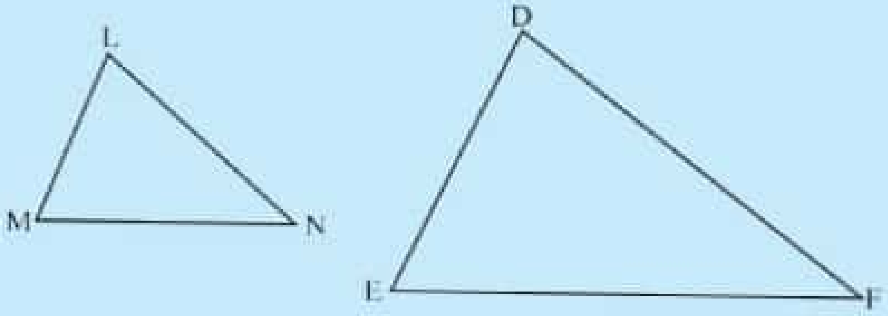
2. SAS Similarity Theorem

If the correspondence between two triangles is such that the lengths of two pairs of corresponding sides are proportional and the included angles are congruent, then the triangles are similar.



3. **SSS Similarity Theorem**

If the correspondence between two triangles is such that the lengths of corresponding sides are proportional, then the triangles are similar.



4. Similar figures have the same shape.

5. In similar figures, the ratios of the lengths of corresponding sides are equal. That is, corresponding sides are proportional. The value of the ratio is called the constant of proportionality or scale factor.

6. The symbol used to indicate similarity between figures is " ~ ".

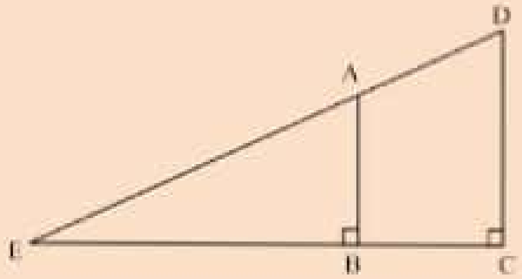
7. Polygons which have all sides congruent and all angles congruent are called regular polygons.

8. If two polygons are similar, the ratio of their perimeter is equal to the ratio of the lengths of any two corresponding sides.

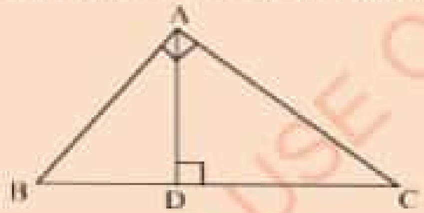
Revision exercise 6

Answer the following questions:

1. In the following figure determine the pairs of equal angles and the proportional sides which make $\Delta AEB \sim \Delta DEC$.



2. Name two triangles similar to ΔABC in the following figure.



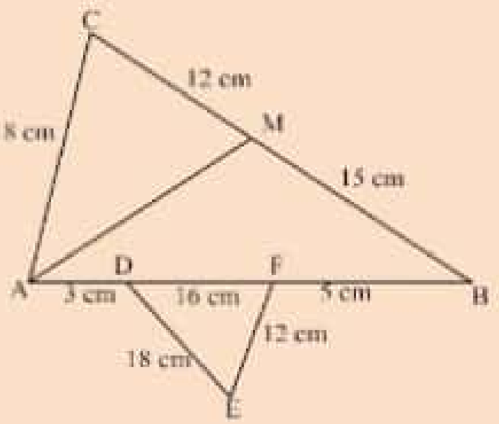
3. If $\Delta LMN \sim \Delta PQR$ and $\Delta PQR \sim \Delta ABC$, mention equal angles in ΔABC and ΔLMN, hence mention all proportional sides in ΔPQR and ΔABC. Find the ratio of proportionality for the similarity of ΔPQR and ΔABC.

4. A triangle ABC is such that $\overline{CA}$ is extended to X and $\overline{BA}$ is extended to Y so that $\overline{XY} // \overline{BC}$. Prove that $\frac{\overline{AX}}{\overline{AC}} = \frac{\overline{XY}}{\overline{CB}}$.

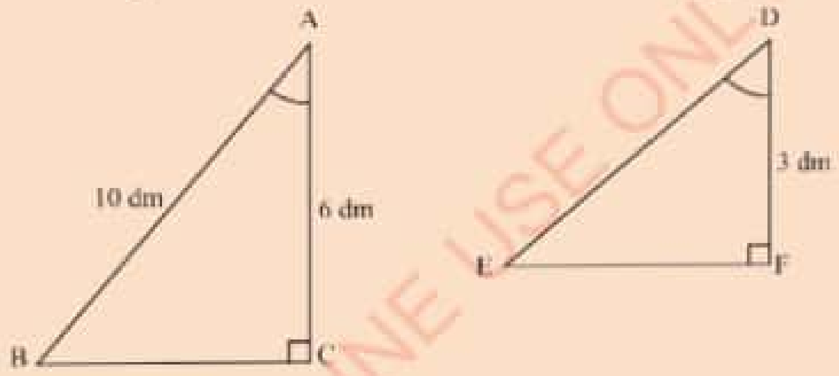
5. A trapezium PQRS is such that $\overline{PQ} // \overline{RS}$. If the diagonals intersect at X, prove that $\frac{\overline{PX}}{\overline{RX}} = \frac{\overline{QX}}{\overline{SX}}$.

Math F2 - (BOOK) (DUMMY) Sept 4.indb 213

6. Name a triangle which is similar to ΔABC in the following figure.



7. Use similarity to find the value of $\overline{ED}$ in the following figures.



8. If $\Delta PQR \sim \Delta LMN$ and $\overline{PR} = 20$ dm, $\overline{NL} = 10$ dm, $\overline{NM} = 12$ dm and $\overline{LM} = 9$ dm, find the lengths of the other sides of ΔPQR.

9. Prove that any two equilateral triangles are similar.

10. In ΔPMT and ΔQNS, $P\hat{M}T = Q\hat{S}N = M\hat{T}P = Q\hat{N}S$.
Prove that $\overline{PM} \times \overline{NS} = \overline{QS} \times \overline{MT}$

Chapter Seven

Geometrical transformations

Introduction

Triangles, circles, and squares are few examples of geometrical shapes. Transforming a given geometrical shape means to make some changes to it. The transformation may either result in change of size of the transformed geometrical shape leaving the shape unchanged, or may not result in change of shape. In this chapter, you will learn about reflection, translation, enlargement, and rotation as forms of linear transformations. Figures in a plane can be translated, reflected, enlarged, or rotated to produce new figures. The competences developed in this chapter will enable you to design, copy, enlarge, and diminish different objects as well as understand and appreciate the works of carpenters, tailors, and building constructors.

Reflection

When you look in a mirror, you see your image on the other side of the mirror. In other words, you see a reflection of yourself. **A reflection** is a transformation which reflects all points of a plane in a line called the mirror – line. The image of an object is as far behind the mirror as the object is in front of the mirror, as seen in Figure 7.1.

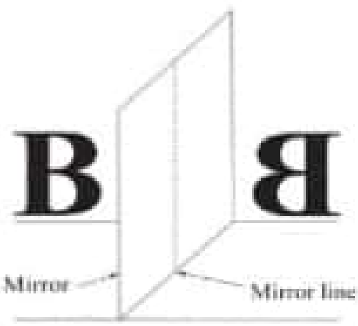


Figure 7.1: *Reflection in a mirror*

Activity 7.1: Drawing an image of a figure in the xy – plane

Materials required: Plane mirrors, pencils, graph paper.

Do the following activity in pairs:

1. (a) Look at yourself through the mirror. How far is your image from the mirror?
   (b) Keeping the mirror in place move backward or forward. What happens by doing these movements?
2. (a) Draw ΔABC in the xy – plane with sides of lengths 6, 8 and 10 centimeters.
   (b) If the triangle ΔABC is reflected in the x – axis, draw its image in the same plane.

Generally, the following are the properties of reflections:

(i) The original object and its image are congruent.

(ii) The original object is reversed by the mirror in the image.

(iii) A line joining any point to its image is perpendicular to the mirror.

(iv) The angle between any line (object) and the mirror line, is equal to the angle between its image and the mirror line. That is, the mirror line bisects the angle formed by the object and the image.

Reflection in the x – axis

The notation M_x is used to denote reflection in the x – axis. Generally, $M_x(a, b)$ represents a reflection of coordinates (a, b) in the x – axis.

When a point is reflected across the x – axis, the x – coordinate remain the same, but the y – coordinate is transformed into its opposite sign. That is, the sign of the y – coordinate is changed. Generally, $M_x(x, y) = (x, -y)$.

Example 7.1

Find the image of the point A(2, 1) after a reflection in the x – axis.

Solution

Locate the coordinates of the point A(2, 1) and its image A′ such that line AA′ that crosses the x – axis at B is perpendicular to it. For reflection, distance AB is equal to distance BA′.

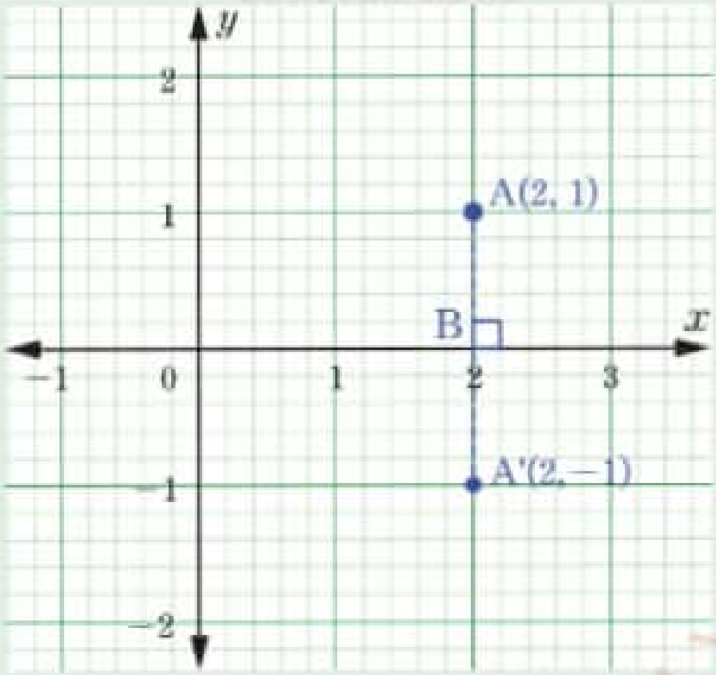


Figure 7.2 *Reflection of a point in the x – axis*

From the Figure 7.2, the coordinates of A′ are (2, –1). The figure shows that the image of A(2, 1) under reflection in the x–axis is A′(2, –1). Since, M_x is used to represent reflection in the x –axis, the above results can be written as $M_x(2, 1) = (2, -1)$.

Example 7.2

The vertices of a triangle are A(1, 2), B(3, 1) and C(-3,2). If the triangle ABC is reflected in the x – axis, find the coordinates of the vertices of its image.

Solution

$M_x(1, 2) = (1, -2)$

$M_x(3, 1) = (3, -1)$

$M_x(-3, 2) = (-3, -2)$

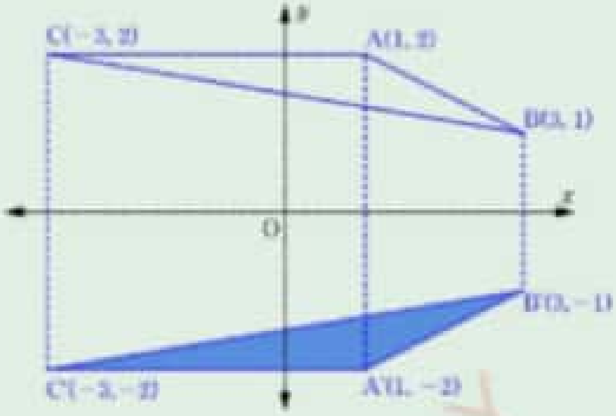


Figure 7.3: *Reflection of a triangle in the x – axis*

Therefore, the coordinates of the vertices of the image are A'(1,-2), B'(3,-1), C'(-3,-2).

Reflection in the y – axis

We use the notation My(*a*, *b*) to represents a reflection of the coordinates (*a*, *b*) in the y – axis.

When a point is reflected across the y – axis, the y – coordinate remain the same, but the x – coordinate is transformed into its opposite sign. That is, the sign of x – coordinate is changed. Generally, $M_y(x, y) = (-x, y)$.

Example 7.3

Find the image of B(3, 4) under a reflection in the y – axis.

Solution

Locate the coordinate of the point B(3, 4) and its image B' such that the line BB' is perpendicular to the y – axis and cross it at C and $\overline{BC} = \overline{CB'}$ as shown in Figure 7.4.

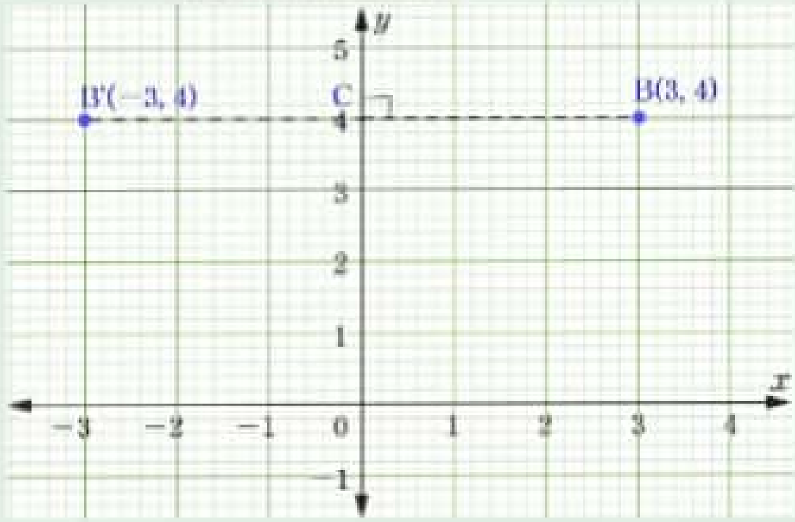


Figure 7.4: *Reflection of a point in the y – axis*

From the figure, the reflection of B(3, 4) in the y – axis is B′(–3, 4).
Therefore, $M_y(3, 4) = (-3, 4)$.

Example 7.4

The vertices of a triangle are P(1, 0), Q(3, 0) and R(1, 2). If the triangle PQR is reflected in the y – axis, find the co-ordinates of the vertices of its image.

Solution

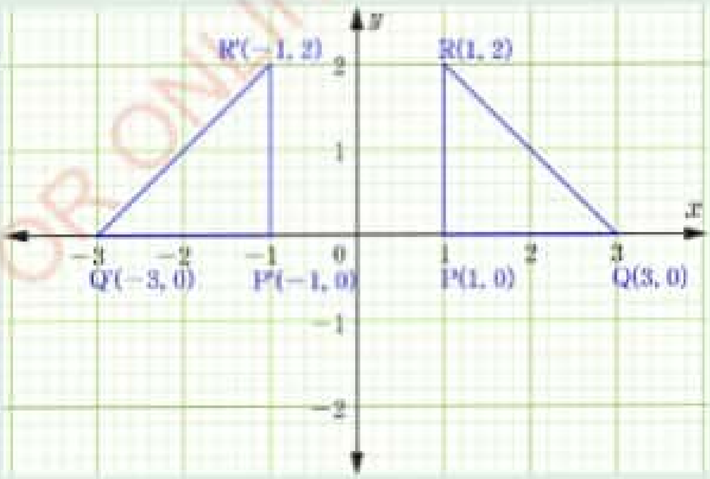


Figure 7.5: *Reflection of a triangle in the y – axis*

Therefore, the coordinates of the vertices of the image are P′(–1, 0), Q′(–3, 0), R′(–1, 2).

Reflection in the line $y = x$

The line $y = x$ makes an angle 45° with the x and y axes. It is the line of symmetry for the angle formed by the two axes. By using the isosceles triangle properties, reflection of the point (1, 0) in the line $y = x$ will be (0, 1).

The reflection of (0, 2) in the line $y = x$ will be (2, 0). In Figure 7.6, it can be observed that the positions of the coordinates are exchanged. Generally, the reflection of the point P(a, b) in the line $y = x$ is P′(b, a).

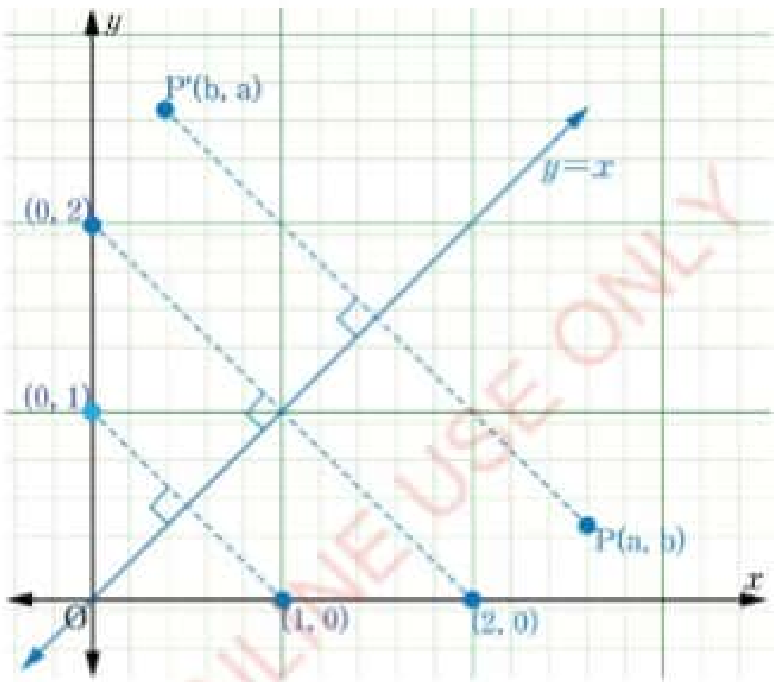


Figure 7.6: *Reflection of points in the line* $y = x$.

Example 7.5

Find the image of the point A(1, 2) after a reflection in the line $y = x$ and draw both the point and its image.

Solution

The image of A(1, 2) in the line $y = x$ is the point A′(2, 1) as shown in Figure 7.7.

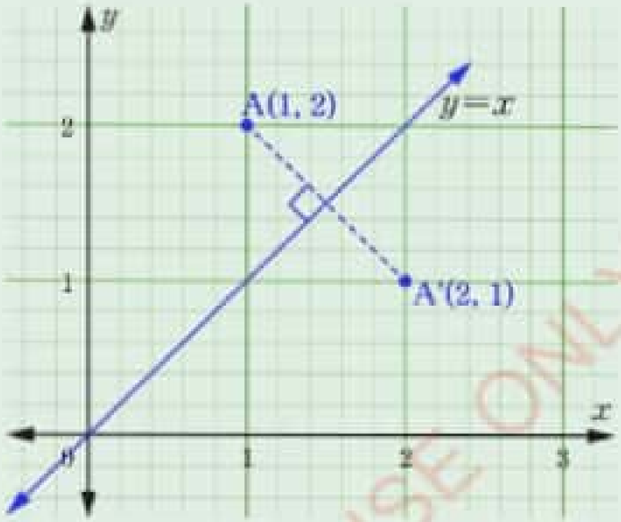


Figure 7.7: *Reflection of the point A(1, 2) in the line y = x*

Reflection in the line $y = -x$

When a point is reflected in the line $y = -x$, the coordinates exchange positions and their signs change as well. Generally, reflection of a point B(c, d) in the line $y = -x$ is B′($-d$, $-c$).

Example 7.6

Find the image of the point A(3, 2) after a reflection in the line $y = -x$ and sketch both the point and its image.

Solution

The reflection of a point in the line $y = -x$ is as shown in Figure 7.8.

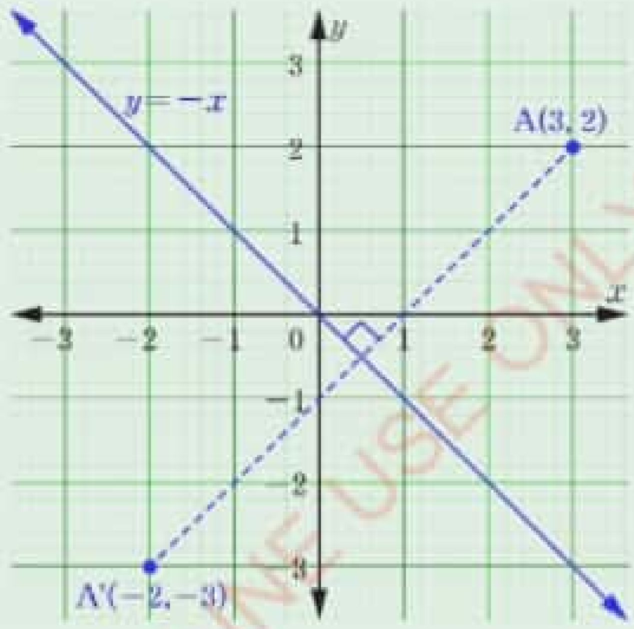


Figure 7.8: *Reflection of a point A(3, 2) in the line y = −x*

Therefore, the reflection of the point A(3, 2) in the line $y = -x$ is $A'(-2, -3)$.

Example 7.7

Find the image of B(3, 4) after a reflection in the line $y = -x$ followed by another reflection in the line $y = 0$. Sketch the point B and the images formed.

Solution

The reflection in the line $y = -x$

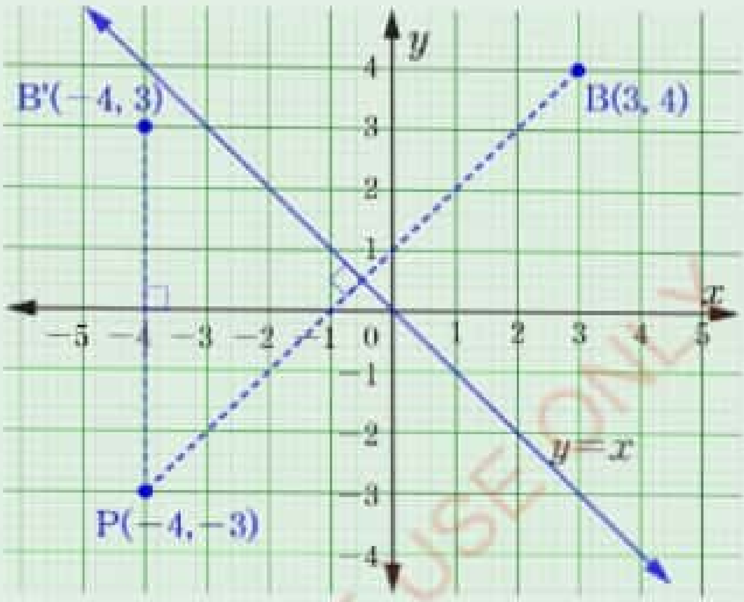


Figure 7.9: *Reflection of a point B(3, 4) in the line $y = -x$ followed by reflection in the line $y = 0$.*

As shown in the Figure 7.9, the image of B(3, 4) after reflection in the line $y = -x$ is B′(−4, −3). The image of B′ (−4, −3) after reflection in the line $y = 0$ (x – axis) is B″(−4, 3).

Therefore, the last image is B″(−4, 3).

Important concepts to remember are the following:

(i) Reflection of $P(x, y)$ in the x – axis or $y = 0$ gives the image $P'(x, -y)$.

(ii) Reflection of $P(x, y)$ in the y – axis $(x = 0)$ gives the image $P'(-x, y)$.

(iii) Reflection of $P(x, y)$ in the line $y = x$ or $y - x = 0$ gives the image $P'(y, x)$.

(iv) Reflection of $P(x, y)$ in the line $y = -x$ or $y + x = 0$ gives the image is $P'(-y, -x)$.

Exercise 7.1

Answer the following questions:

1. Find the image of the point D(4, 2) under a reflection in the x – axis.
2. Find the image of the point P(–2, 5) under a reflection in the y – axis.
3. Point Q(–4, 3) is reflected in the y – axis. Find the coordinates of its image.
4. Point R(6, –5) is reflected in the y – axis. Find the co-ordinates of its image.
5. Reflect the point (1, 2) in the line $y = -x$.
6. Reflect the point (5, 3) in the line $y = x$.
7. Find the image of the point (1, 2) after a reflection in the line $y = x$ followed by another reflection in the line $y = -x$.
8. Find the image of the point P(–2, 1) in the line $y = -x$ followed by another reflection in the line $x = 0$. Sketch the positions of the image P′ and the point P, indicating clearly the lines involved.
9. Find the coordinates of the image of the point A(5, 2) under a reflection in the line $y = 0$.
10. Find the coordinates of the image of the point B (–6, 5) under a reflection in the line $x = 0$.
11. The coordinates of the image of a point R reflected in the x – axis is R′(2, –9). Find the coordinates of R.
12. The vertices of the triangle PQR are P(6, 2), Q(–2, 8), and R(–5, –11). If triangle PQR is reflected in the y – axis, find the coordinates of the vertices of its image.
13. The vertices of a polygon are A(2, 3), B(2, –4), C(–4, –4), and D(4, 3). If the polygon ABCD is reflected in the y – axis:
    (a) Find the coordinates of its image.
    (b) Draw a sketch to show the image.
14. Describe the transformation which moves the point (4, 2) to (–4, 2) on the xy – plane.
15. The point (6, –3) is moved by a transformation T to the point (6, 3). What is T?

Rotation

Rotation is a transformation which turns an object through a certain angle. Rotation can be either in clockwise or anticlockwise direction. When an object is rotated about a fixed point, this means that all points of the object are rotated. In order to describe a rotation, consideration is on the centre of rotation, the angle of rotation and the direction of rotation.

The transformation of rotation is usually denoted by R. The symbol R_θ means that an object is rotated through an angle θ in the xy–plane, the value of θ is negative when rotated in the clockwise direction, and is positive when it is rotated in the anticlockwise direction.

Activity 7.2: Drawing angles

Materials required: Plain papers, protractor, ruler, pencil

Draw an angle for each of the following degree measures:

(a) 45°   (b) 60°   (c) 105°   (d) 150°

Properties of rotation

(i) The original object and its image are congruent.

(ii) The original object and its image have the same orientation.

(iii) The centre of rotation is the only fixed (invariant) point, all other points change their positions.

(iv) Every point turns through the same angle.

Example 7.8

Find the image of the point P(1, 0) after a rotation through 90° about the origin in the anticlockwise direction.

Solution

The rotation of P(1, 0) through 90° anticlockwise about the origin is as shown in Figure 7.10.

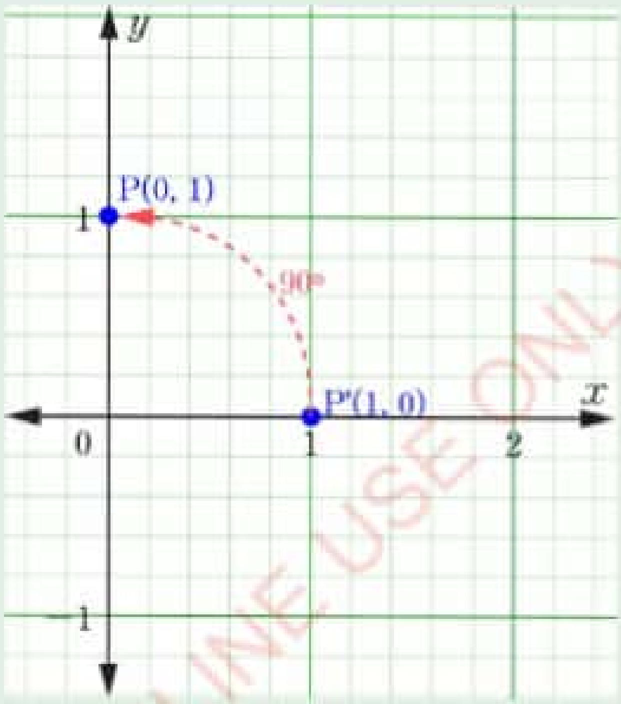


Figure 7.10: *Rotation of a point through 90°*

After rotation of a point through 90° about the origin, the image will be on the y – axis. Since P is 1 unit from O on the x – axis, and P' is 1 unit from O on the y – axis. The image of P is P' (0, 1) as shown in Figure 7.10.

Therefore, $R_{90^\circ}(1,0)=(0,1)$.

Generally, rotation of a point on the x-axis through 90° anticlockwise about the origin is given by $R_{90^\circ}(x,0)=(0,x)$.

Example 7.9

Find the image of the point B(4, 2) after a rotation through 90^0 about the origin in the anticlockwise direction.

Solution

Rotation of point B(4, 2) through 90^0 anticlockwise about the origin is as shown in Figure 7.11.

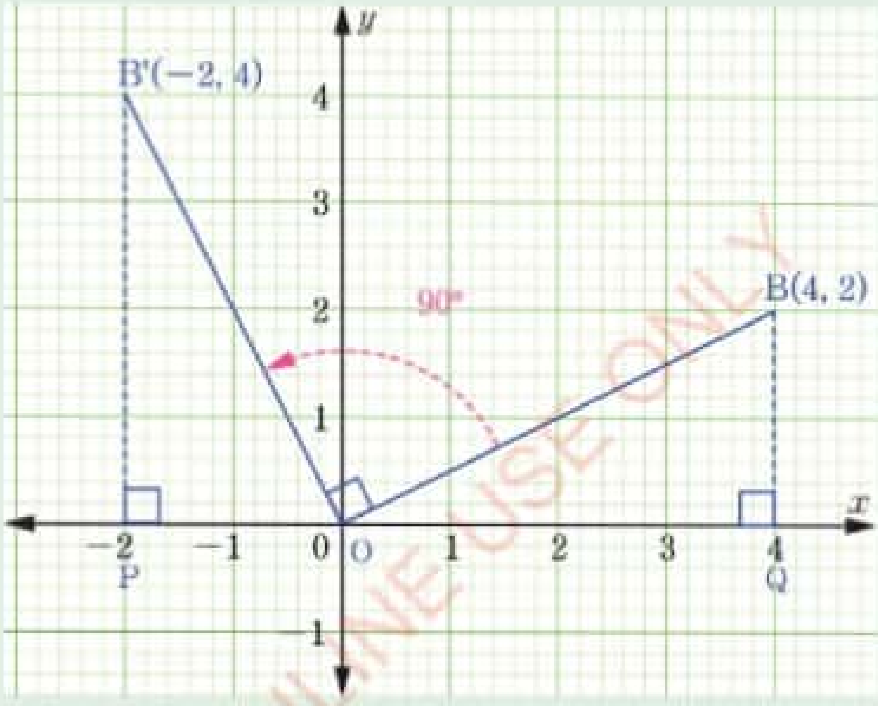


Figure 7.11: *Rotation of point B(4, 2) through 90° anticlockwise about the origin*

From figure 7.11, $\overline{OB} = \overline{OB'}$, $Q\hat{B}O = P\hat{O}B'$ and $Q\hat{O}B = P\hat{B}'O$

$\Delta PB'O \equiv \Delta QOB$

Hence, $\overline{PB'} = \overline{QO}$

$R_{90^0}(4,2) = (-4, 2)$.

Therefore, $R_{90^0}(4,2) = (-2, 4)$

Example 7.10

Find the image of the point H(1, 3) after a rotation through 180° about the origin in the anti–clockwise direction.

Solution

The rotation through 180° about the origin is as shown in Figure 7.12.

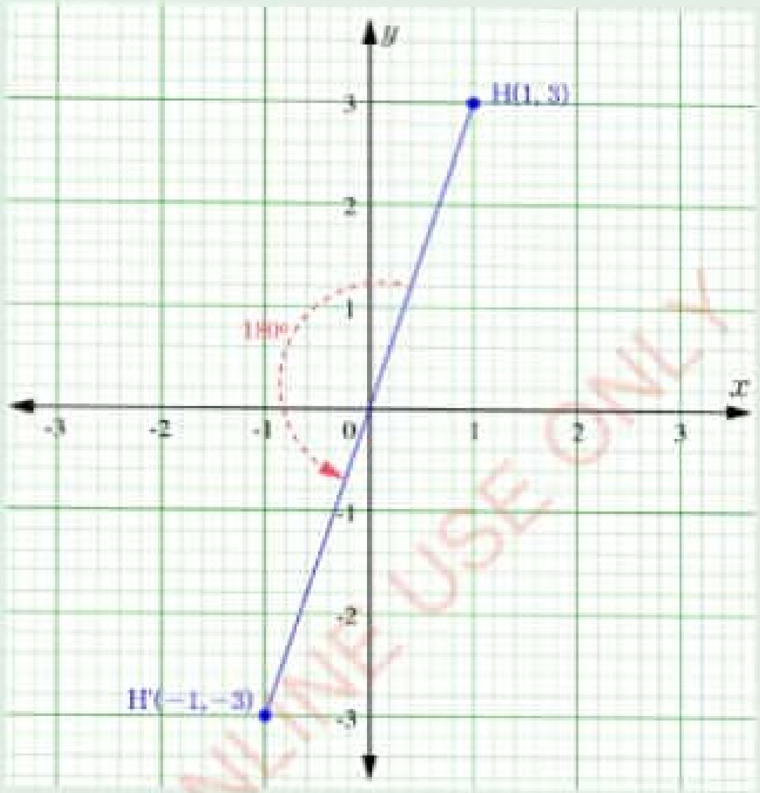


Figure 7.12: *Rotation of a point through 180° about the origin*

From the graph, $R_{180°}(1, 3) = (-1, -3)$

Generally, rotation of a point through 180° anticlockwise about the origin is given by $R_{180°}(x, y) = (-x, -y)$

Example 7.11

Find the image of the point Q(2, 1) after a rotation through $270°$ about the origin in the anticlockwise direction.

Solution

The rotation of a point Q(2, 1) through 270° anticlockwise about the origin is shown in Figure 7.13.

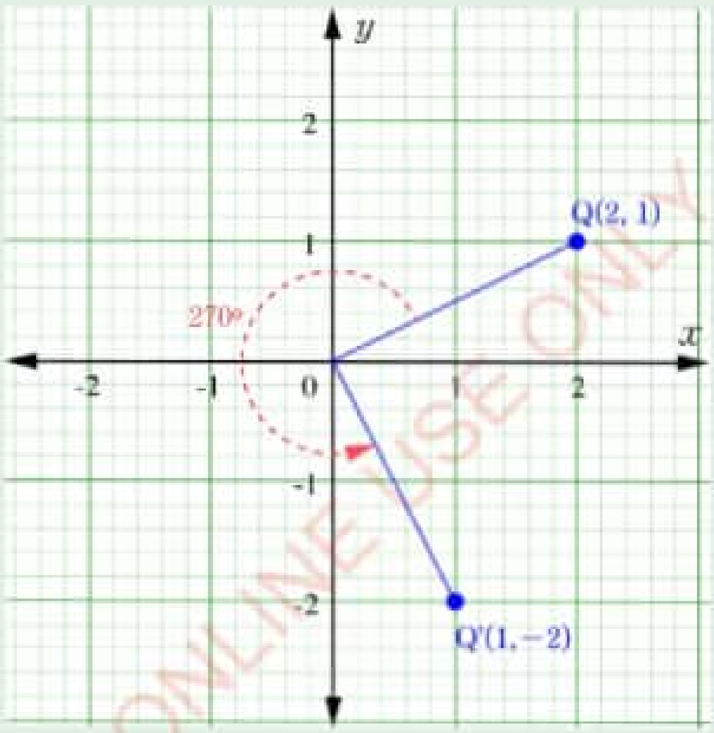


Figure 7.13: *Rotation of a point through 270° anticlockwise about the origin*

From Figure 7.12, $R_{270°}(2, 1) = (1, -2)$

Generally, the rotation of a point through 270° anticlockwise about the origin is given by $R_{270°}(x, y) = (y, -x)$.

Example 7.12

Find the image of the point P(2, 4) after a rotation through 90° about the origin in the clockwise direction.

Solution

The rotation of a point through 90° clockwise about the origin is as shown in Figure 7.14.

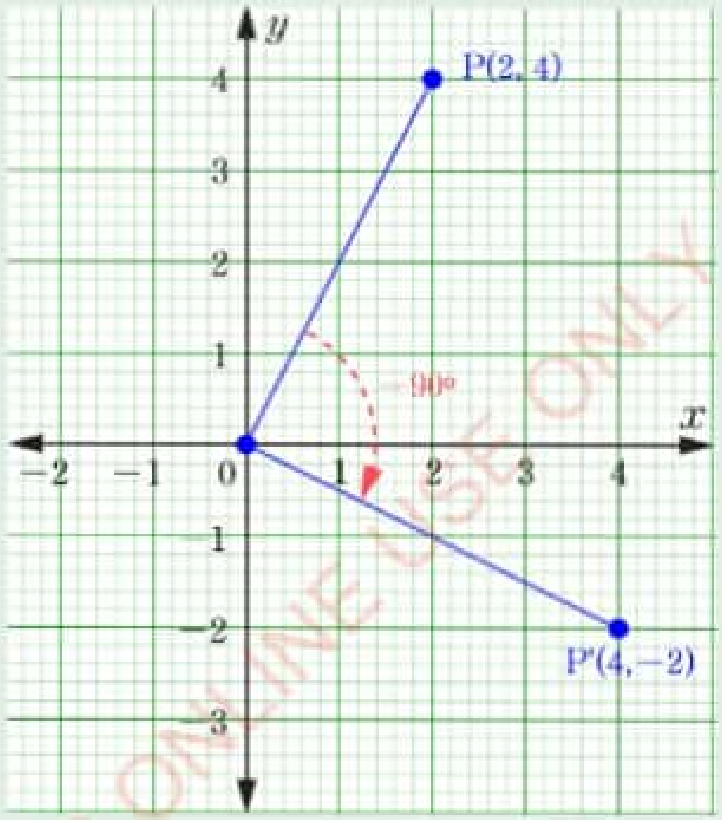


Figure 7.14: *Rotation of a point through 90° clockwise about the origin*

From the graph in Figure 7.13, $R_{90°}(2, 4) = (4, -2)$.

Generally, the rotation of a point through 90° clockwise about the origin is given by $R_{-90°}(x, y) = (y, -x)$.

Example 7.13

Find the image of the point L(1, 2) under a rotation through 180° clockwise about the origin.

Solution

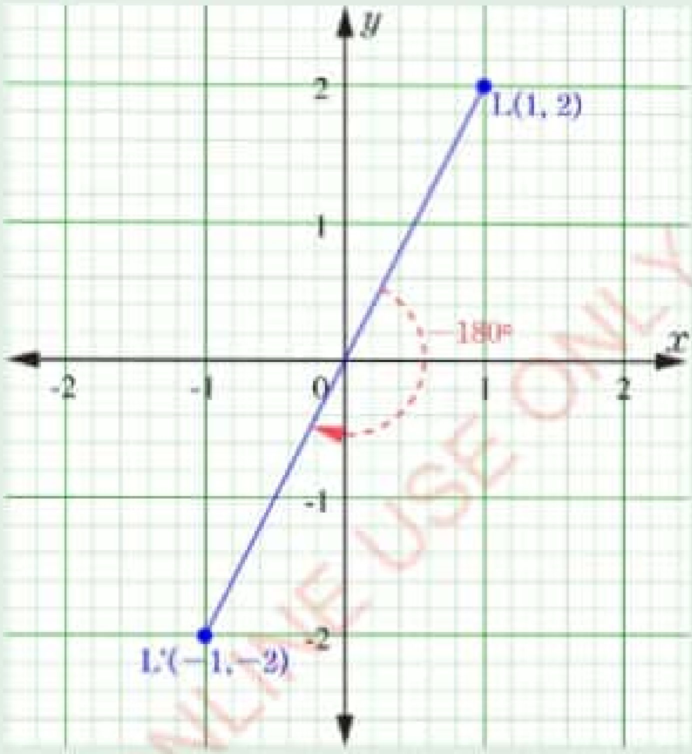


Figure 7.14: *Rotation of a point through 180° clockwise about the origin*

From the graph in Figure 7.14, $R_{-180°}(1, 2) = (-1, -2)$

Generally, the rotation of a point through 180° clockwise about the origin is given by $R_{-180°}(x, y) = (-x, -y)$.

Example 7.14

Find the image of the point Q(3, 2) after rotating through 270° clockwise about the origin.

Solution

The rotation of a point through 270° clockwise about the origin is as shown in Figure 7.16.

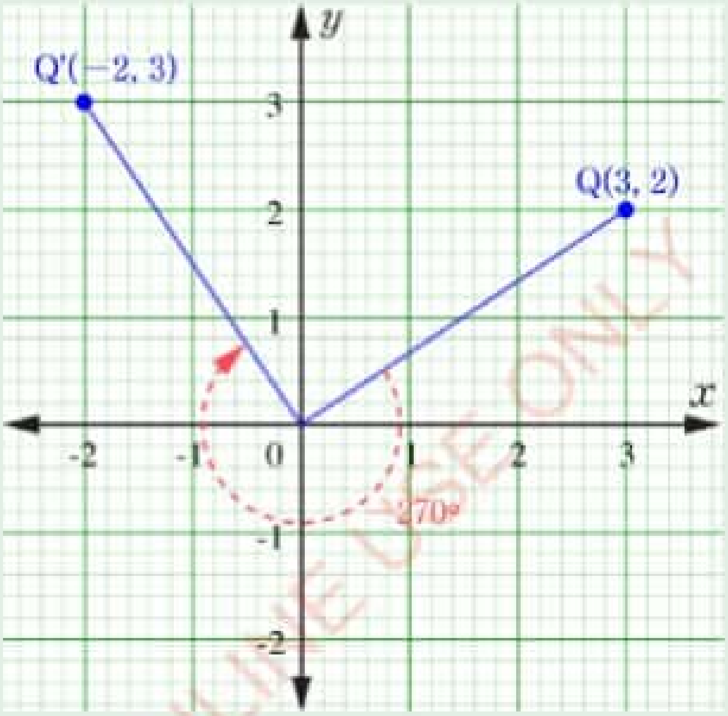


Figure 7.16: *Rotation of a point through 270° clockwise about the origin*

From the graph in Figure 7.16, $R_{-270°}(3, 2) = (-2, 3)$.

Generally, the rotation of a point through 270° clockwise about the origin is given by $R_{-270°}(x, y) = (-y, x)$.

Activity 7.3: Finding the image of a point rotated about the origin through the given angle and direction.

In pairs or groups, complete the following table for rotation of each point about the origin:

Point	90° Clockwise	180° Anticlockwise	270° Anticlockwise	270° Clockwise
(5, 2)				
(−4, 3)				
(−3, −2)				
(3, 1)				
(0, 5)				
(−6, 0)				

Summary

Rotation about origin	Anticlockwise $+\theta$	Clockwise $-\theta$
$R_{90°}(x, y)$	$(-y, x)$	$(y, -x)$
$R_{180°}(x, y)$	$(-x, -y)$	$(-x, -y)$
$R_{270°}(x, y)$	$(y, -x)$	$(-y, x)$

Exercise 7.2

Answer the following questions:

1. Find the image of the point (1, 2) under a rotation through 180° anti-clockwise about the origin.
2. Find the image of the point (6, 0) under a rotation through 90° clockwise about the origin.
3. Find the image of the point (−2, 1) under a rotation through 270° clockwise about the origin.
4. Point Q(5, −4) is rotated through 270° in the clockwise direction about the origin. Find the coordinates of its image.

5. Find the image of (1, 2) after a rotation through -90° about the origin.
6. Find the image of $(-3, 5)$ after a rotation through -180° about the origin.
7. Find the image of $(-5, 0)$ after a rotation through -180° about the origin.
8. Find the image of $(-5, 0)$ after a rotation through 180° about the origin. Comment about the results in questions 7 and 8.
9. The vertices of triangle OAB are O(0, 0), A(2, 3) and B(2, 1). The triangle is rotated through 90° anticlockwise about the origin. Find the co-ordinates of its image.
10. The vertices of rectangle PQRS are P(0,0), Q(3, 0), R(3, 2) and S(0, 2). The rectangle is rotated through 90° clockwise about the origin.
    (a) Find the co-ordinates of its image.
    (b) Draw the image.

Translation

A translation is a geometrical transformation that moves each point of an object or a figure by the same distance in a given direction. This means that, under translation, a figure or an object moves from one place to another without changing its size, arrangement, or orientation. When an object undergoes a translation, all its points move on a plane through the same distance in the same direction.

Activity 7.4: Identifying changes on an object being translated.

Individually or in pairs perform the following tasks:
Put your Mathematics textbook on a flat table. Mark the corner points on the table, push it 10 cm horizontally to the right from the first position, then answer the following questions:

(a) Did the textbook change its orientation?
(b) How far did each corner of the book move?
(c) Did the size of the book change?

In Figure 7.17, ΔLMN slides to ΔL'M'N' in the direction of NN'. Note that lines LL', NN' and MM' are parallel and are of equal length. The triangle LMN is mapped onto L'M'N' by a translation.

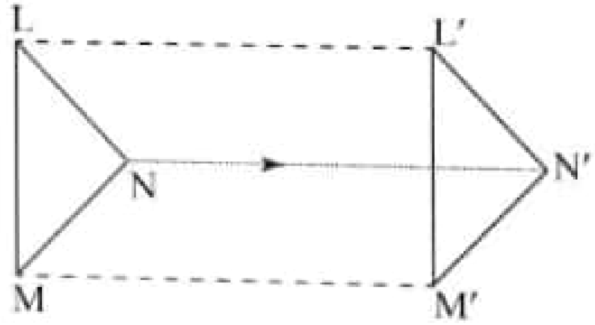


Figure 7.17: *Translation of triangle LMN*

Example 7.15

In Figure 7.18, move each point of the coloured object 5 units to the right and 3 units downwards.

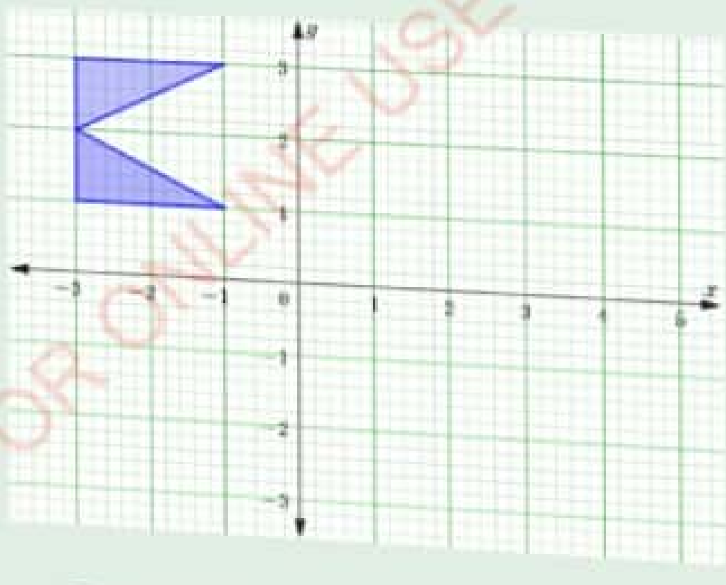


Figure 7.18: *An object in the xy – plane*

Solution

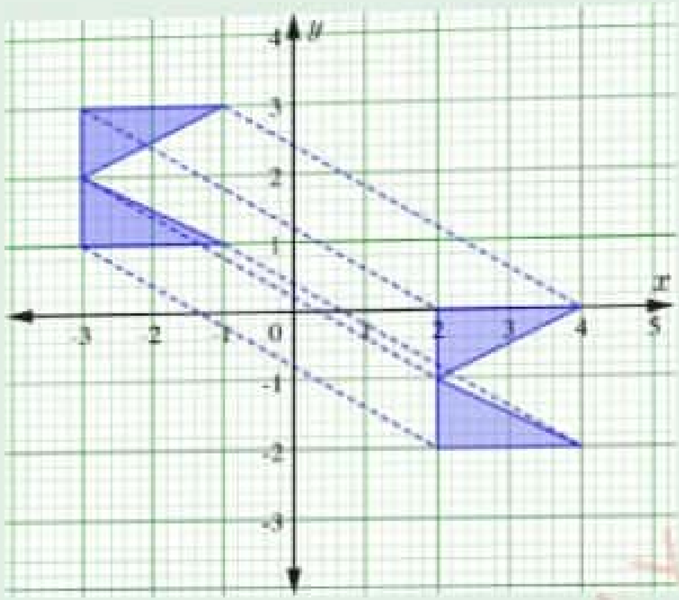


Figure 7.19: *Translation of an object to the right and then downwards*

The following are the properties of translations:

(i) The original object and its image are congruent.

(ii) The original object and its image have the same orientation.

(iii) The lines drawn from any point on the object to their corresponding points on the image are all parallel and equal in length.

(iv) The corresponding sides and angles are equal.

Note: A translation usually is denoted by the letter T.

For example;

T(1, 1) = (6, 1) means that the point (1, 1) has been moved to (6, 1) by a translation T. This translation will move the origin from (0, 0) to (5, 0). This is written as T(0, 0) = T(5, 0).

Example 7.16

A translation takes the origin to (–2, –5). Find where it takes (2, –3).

Solution

The translation where it takes (2, –3) from origin is as shown in Figure 7.20.

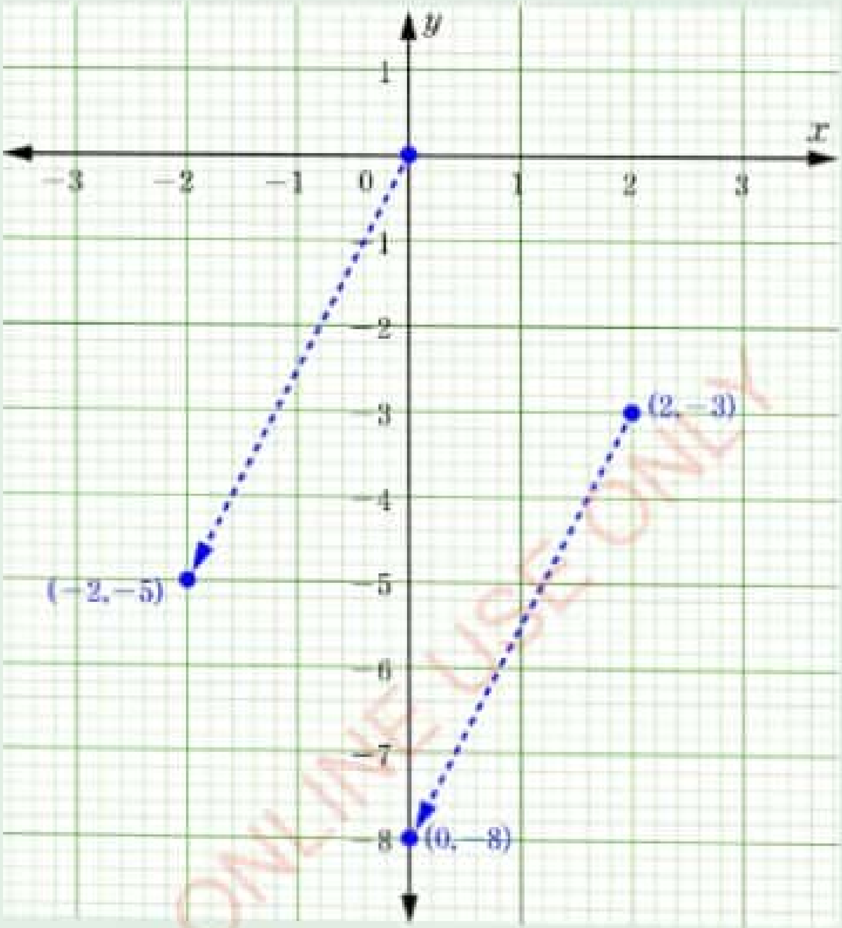


Figure 7.20: *The image of (2, –3) by the translation (–2, –5)*

Point O is moved 2 units to the left along the x–axis and 5 units along the y–axis in the negative direction.

Point (2, –3) must be moved 2 units to the left along the x–axis and 5 units along the y–axis in the negative direction. So, results to $2 - 2 = 0$ for the x–coordinate and $-3 - 5 = -8$ for the y–coordinate.

Therefore, the translation takes the point (2, –3) to (0, –8).

Example 7.17

The square object CDEF in Figure 7.21 is the image of ABCD under a reflection in CD.

(a) What translation maps ABCD onto DCFE?

(b) What is the image of AB by reflection in DC?

(c) What is the image of AC by reflection in AD?

Solution

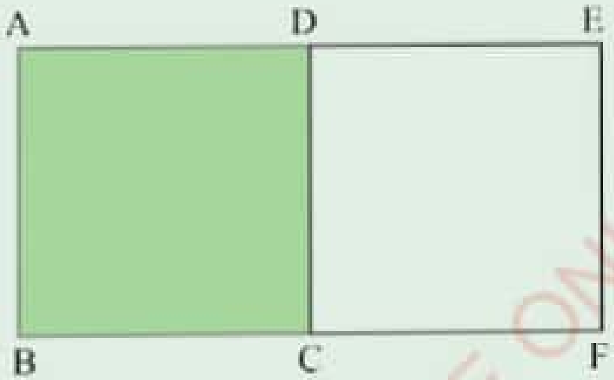


Figure 7.21: *Reflection of an object ABCD*

(a) $\overline{BC} = \overline{CF}$

BC has moved a distance CF in the direction from B to C.

Therefore ABCD is moved to DCFE by the translation through distance BC.

(b) A moves a distance AD to D and B moves a distance BC to C.

Therefore, the image of AB is DC.

(c) D moves a distance AD to E and C moves a distance BC to F.

Therefore, the image of DC is EF.

Activity 7.5: Drawing images of figures

Draw the images of figures in the xy – plane by following the given steps:

Materials required: Sheets of graph papers, a ruler and a pencil.

Steps:

1. Locate each of the points in parts (a), (b) and (c) on a separate graph paper.

   (a) A (2, 1), B (3, 5), C (7, 0)
   $(x, y) \rightarrow (x + 2, y + 4)$

   (b) D(–1, 3), E(–2, 5), F(–2, 0)
   $(x, y) \rightarrow (x - 4, y - 2)$

   (c) R(2, 2), S(5, –1), F(–2, 0), T(–3, –4)
   $(x, y) \rightarrow (x + 6, y - 3)$

2. Join the points located on each graph paper to form a figure. Note that the lines should not cross each other when joining the points in part (c).
3. Use the given transformation to obtain the image of the points for each figure, then plot and join the image points in the respective graph paper.
4. For parts (a), (b) and (c) compare the distances between the original points and their corresponding images.

Exercise 7.3

Answer the following questions:

1. A translation takes the origin to (–2, 5). Find where it takes:
   (a) (–6, 6) (b) (5, 4)
2. A translation takes every point a distance 1 unit to the left and 2 units downwards on the xy-plane. Find where it takes:
   (a) (0, 0) (b) (1, 1) (c) (3, 7)
3. A translation takes the point (3, 2) to (–4, –5). Find where it takes:
   (a) (0, 0) (b) (5, 5)

4. A translation moves the origin a distance 2 units along the line $y = x$ upwards. Find where it takes the following points:

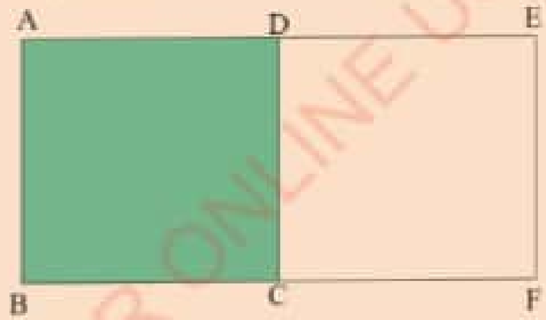
(a) (0, 0) (b) (2, –1) (c) (1, 1)

5. A rectangular object ABCD is shown in the following figure.

(a) What is the image of D by reflection in AB?

(b) What is the image of A by reflection in BC?

6. In the following figure, the square CDEF is the image of ABCD by line of symmetry in CD.



(a) What translation maps Δ BCD onto Δ CFE?

(b) What is the image of BD by translation AD?

7. A translation moves the origin to (h, k). Where does it move the point (x, y)?

8. Draw the triangle whose vertices are A(2, –2), B(6, 1) and C(–1, 5). Find and draw the image of a triangle formed by the translation which moves the origin to (1, 3).

9. Draw a triangle whose vertices are A(2, –2), B(6, 10) and C(–1, 5). Find and draw the image of the triangle formed by the translation which moves the origin to (1, 3) on the *xy* – plane.

10. If Q (2, 3) is translated by (5, 0), draw an arrow on the *xy*-plane to show the point and its image under the given transformation.

11. If S(8, –2) is translated by (–6, 2), draw an arrow on the *xy*-plane to show the point and its image under the given transformation.

Enlargement

Enlargement or dilation is a transformation in which a figure is made larger or smaller. A photograph may be enlarged to suit a certain purpose as shown in Figure 7.22.

Figure 7.22: *Enlargement of an object*

An original figure can be enlarged by drawing the figure using a scale. Enlarged shapes are **geometrically similar** and their corresponding angles are equal. The number that magnifies a figure is called the **enlargement factor** and is usually denoted by k. Enlargement of a figure is a transformation called a **dilation.** In the case of closed figures, if the lengths are enlarged by a factor k then the area is enlarged by a factor of k^2.

Activity 7.6: Finding the constant of proportionality

Materials required: Manila paper, pencil, ruler, protractors, pairs of scissors

Steps:

1. Draw two similar figures of different sizes.
2. Enlarge the two figures by using different factors. Compare the figures obtained.
3. Obtain the enlargement factor for each pair of corresponding figures.
4. Take a piece of paper and draw two circles of different diameters. Make cuts of the two circles to obtain two circular objects. Are the two objects similar?

Note the following about enlargement:

1. When the enlargement factor is 1, that is $k=1$, the object and its image are equal.
2. When $k>1$ the image will be larger than the original object. In this case the image is enlarged.
3. If $0<k<1$, the image will be smaller than the original object. In this case, the image is said to be diminished.

Example 7.18

A shadow of an object is cast onto a wall. Find the position of the object so that the shadow doubles its size.

Solution

$\Delta OAM \sim \Delta OCN$

$$\frac{\overline{OM}}{\overline{ON}} = \frac{\overline{AM}}{\overline{CN}} = \frac{\overline{OA}}{\overline{OC}} = k = \frac{1}{2}$$

But $\overline{CN} = 2\,\overline{AM}$ (given)

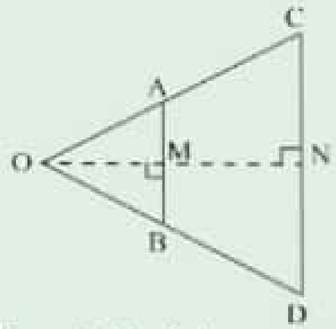


Figure 7.23: *Enlargment of ΔAOB of ΔCOD*

This implies that, $\dfrac{\overline{AM}}{2\overline{AM}} = \dfrac{1}{2}$.

$\frac{1}{2}$ is the similarity ratio of the sides of triangle OAM to triangle OCN

Therefore, $\overline{OM} = \frac{1}{2}\overline{ON}$.

The object should be placed halfway between the source and the shadow. In this example, O is the centre of enlargement.

Example 7.19

In Figure 7.24, rectangle ABCD is enlarged to rectangle AB′C′D′ where A is the centre of enlargement.

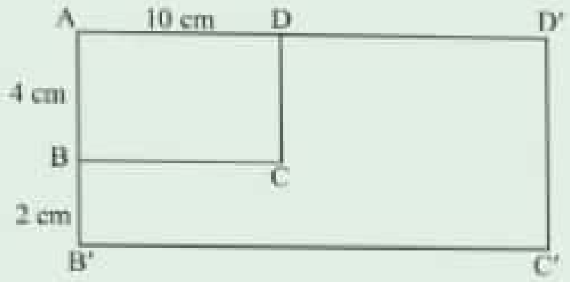


Figure 7.24: *Enlargement of rectangle ABCD to rectangle AB′C′D′*

(a) Find the enlargement factor of the enlargement, given that $\overline{AB} = 4\text{cm}$ and $\overline{BB'} = 2\text{cm}$.

(b) Find length AD′, given that $\overline{AD} = 10\text{cm}$.

Solution

(a) Scale factor $= \dfrac{\overline{AB'}}{\overline{AB}} = \dfrac{6}{4} = 1.5$

Therefore, the scale factor is 1.5.

(b) $\overline{AD'} = \overline{AD} \times$ scale factor

$= 10 \times 1.5$ cm

Therefore, $\overline{AD}$ = 15 cm.

Properties of enlargement

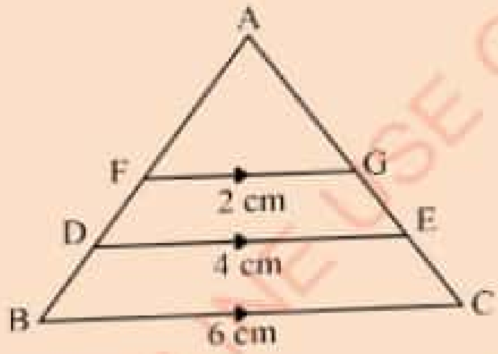
(i) The ratios of the lengths of the sides of the object and their corresponding ratios of images is constant.

(ii) Corresponding angles of the object and its image under enlargement are always equal.

Exercise 7.4

Answer the following questions:

1. Given a triangle PQR with vertices P(0, 0), Q(0, 3) and R(3, 0), draw a triangle P'Q'R' with an enlargement factor of 2.
2. A rectangle ABCD is 4 cm long and 3 cm wide. Draw a similar rectangle PQRS whose length is 1.5 times that of ABCD.
3. The shadow of a circular object whose diameter is 10 cm is 48 cm from a light source. What will be the diameter of the shadow when the object is placed 36 cm from the light source?
4. The line segments $\overline{FG}$, $\overline{DE}$, and $\overline{BC}$ in the following figure are parallel. What is the enlargement factor for transforming:
   (a) ΔADE to ΔABC?
   (b) ΔAFG to ΔADE?

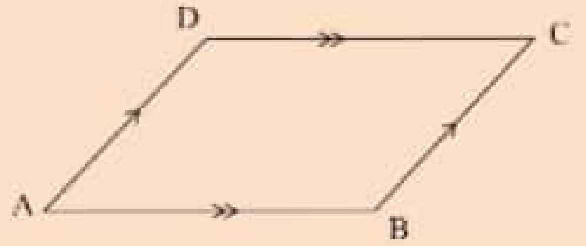


5. A line segment from the origin has its end point at (6, 2). If the line segment is enlarged by a factor of 4, what is the new end point?
6. The line segment AB with coordinates A(4, 0) and B(0, 3) is enlarged to A'B' by a factor of 2. Find the coordinates for A' and B'.
7. Find the image of a circle of radius one unit having its centre at (1, 1) under an enlargement transformation by a factor of 5. Hence, draw the circle and its image on the same xy – plane.
8. Find the image of the square with vertices O(0, 0), A(1, 0), B(1, 1) and C(0, 1) under the enlargement factor of 4. Hence draw both the object and image on the same pair of axes.

9. If the following figure is a parallelogram,

   (a) what is the image of $\overline{BC}$ by the translation through distance $\overline{CD}$?

   (b) name the translation that maps $\overline{AB}$ onto $\overline{DC}$.



Scale of enlargement

The idea of similarity can be used in enlarging geometrical figures. For example, in maps, a large area of land is represented by a small area on a piece of paper using a scale. Scale is a ratio between the measurement of a drawing and the actual measurement. It is normally stated in the form of 1: n. For example, if the scale of a map is 1:20 000, then 1 unit on the map represents 20 000 units on the ground. Now, consider the following two triangles:

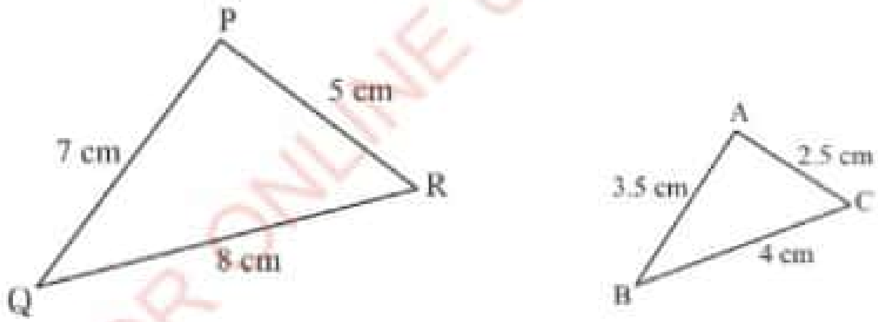


Figure 7.25: *Illustration of scale of enlargement*

In Figure 7.25, a triangle ABC is a scaled version of triangle PQR, where the scale is 1:2. In ratio form, scale is expressed as follows:

Measurement of drawing: Actual Measurement, that is,

$$\text{Scale} = \frac{\text{Measurement of a drawing}}{\text{Actual measurement}}.$$

Normally, the measurement on a drawing is given in centimetres (cm).

Example 7.20

Find the length of a drawing that represents:

(a) 15 km when the scale is 1cm:500 000 cm

(b) 45 km when the scale is 1 cm to 900 m.

Solution

(a) The scale of 1:500 000 means 1 cm on the drawing represents 500 000 cm of length.

First, change 15 km into centimetres

That is 15 km $= 15 \times 100\,000$ cm

$= 1\,500\,000$ cm

Next, find the scale of drawing:

$$\text{Scale} = \frac{\text{Measurement of a drawing}}{\text{Actual measurement}}.$$

$$\frac{\text{Measurement of a drawing}}{1\,500\,000} = \frac{1}{500\,000}$$

$$\text{Measurement of a drawing} = \frac{1}{500\,000} \times 1\,500\,000 \text{ cm} = 3 \text{ cm}$$

Therefore, 15 km is represented by 3 cm.

(b) The scale is 1 cm represents 900×100 cm $= 90\,000$ cm

Scale is 1:90 000

Actual length $= 45$ km $= 45 \times 100\,000$ cm $= 4\,500\,000$ cm

$$\text{Scale} = \frac{\text{Measurement of a drawing}}{\text{Actual measurement}}$$

$$\frac{\text{Measurement of a drawing}}{4\,500\,000} = \frac{1}{90\,000}$$

$$\text{Measurement of a drawing} = \frac{1}{90\,000} \times 4\,500\,000 \text{ cm} = 50 \text{ cm}.$$

Therefore, 45 km is represented by 50 cm.

Example 7.21

Find the actual length in metres represented by:

(a) 3.5 cm when the scale is 1:5 000

(b) 1.8 mm when the scale is 1cm to 500 metres

Solution

(a) Scale is 1:5000, which means 1 cm on the drawing represents 5000 cm of length.

Length of the drawing = 3.5 cm

$$\text{Scale} = \frac{\text{Measurement of a drawing}}{\text{Actual measurement}}$$

$$\frac{3.5}{\text{Actual measurement}} = \frac{1}{5\ 000}$$

$$\begin{aligned}\text{Actual measurement} &= 3.5 \times 5\ 000\\ &= 17\ 500 \text{ cm}\\ &= \frac{17\ 500 \text{ m}}{100}\\ &= 175 \text{ m}\end{aligned}$$

Therefore, the actual length represented by 3.5 cm is 175 m.

(b) The scale is 1 cm to 500 m. But 500 m = 500 × 100 cm = 50 000 cm

Therefore, the scale of the drawing is 1:50 000

Scale = 1:50 000

Length of drawing = 1.8 mm = 0.18 cm

$$\text{Scale} = \frac{\text{Measurement of a drawing}}{\text{Actual measurement}}$$

$$\frac{0.18}{\text{Actual measurement}} = \frac{1}{50\ 000}$$

$$\text{Actual measurement} = 0.18 \times 50\ 000\,\text{cm}$$
$$= 9\ 000\ \text{cm}$$
$$= \frac{9\ 000\ \text{m}}{100}$$
$$= 90\ \text{m}$$

Therefore, the actual length represented by 1.8 mm is 90 m.

Scale factor

If two polygons are similar and the ratio of their corresponding sides is $a:b$ then the scale factor of enlargement is $\frac{a}{b}$. For example, the corresponding sides of two similar polygons have the ratio 5:3, then the scale of enlargement is $\frac{5}{3}$.

Example 7.22

From Figure 7.26, find the scale factor of enlargement, of ΔXYZ and ΔXMN then calculate $\overline{NM}$.

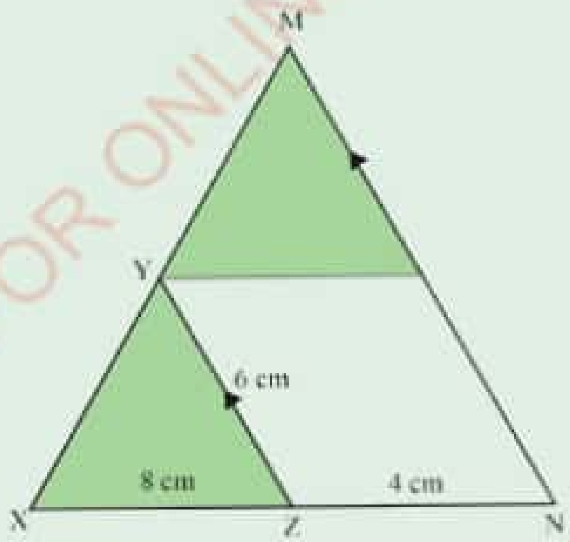


Figure 7.26: *Illustration of scale factor*

Solution

Triangle XYZ and triangle XMN are equiangular.

$\Delta XYZ \sim \Delta XMN$

$$\frac{\overline{XN}}{\overline{XZ}} = \frac{\overline{XM}}{\overline{XY}} = \frac{12\text{cm}}{8\text{cm}} = \frac{3}{2}$$

Therefore, the scale of enlargement is $\frac{3}{2}$.

Also, $\frac{\overline{XM}}{\overline{XY}} = \frac{\overline{NM}}{\overline{ZY}} = \frac{3}{2}$

$$\frac{\overline{NM}}{6} = \frac{3}{2}$$

$$\overline{NM} = 6 \times \frac{3}{2}\text{cm} = 9 \text{ cm}.$$

Therefore, $\overline{NM} = 9$ cm.

Scale factor for areas

Consider the two similar rectangles shown in Figure 7.27 with a scale factor k.

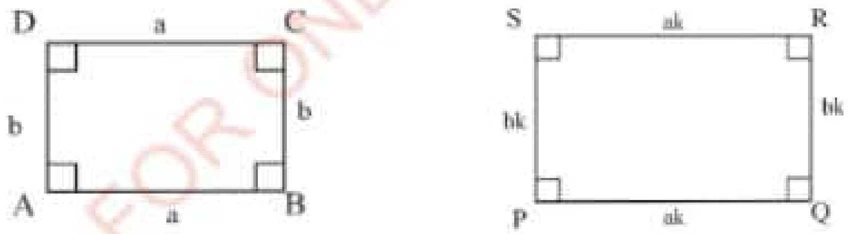


Figure 7.27: *Similar rectangles*

If AB = a and AD = b, then PQ = ak and PS = bk.

$$\text{Area of ABCD} = a \times b = ab$$

$$\text{Area of PQRS} = ak \times bk = abk^2$$

$$\frac{\text{Area of PQRS}}{\text{Area of ABCD}} = \frac{abk^2}{ab} = k^2$$

Therefore, if two polygons have a scale factor k, then the ratio of their areas is k^2. This is also called the scale factor for the area.

Example 7.23

In Figure 7.28 $\Delta ABC \sim \Delta STU$, with $\overline{AB} = 3$ cm, $\overline{ST} = 2$ cm, and the area of ΔSTU is 6 cm^2. Find the area of ΔABC.

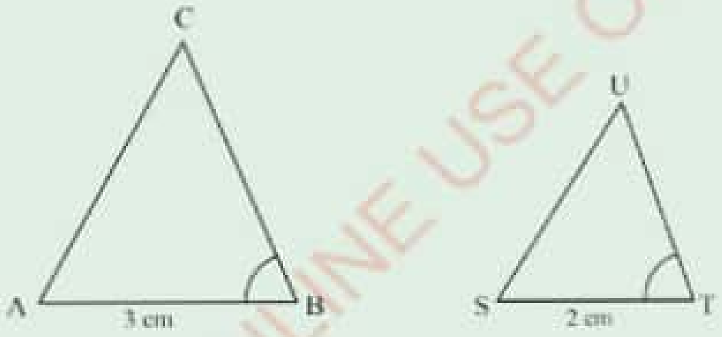


Figure 7.28: *Area of triangles by ratios of their lengths*

Solution

$$\text{The scale factor} = \frac{\overline{AB}}{\overline{ST}} = \frac{3}{2}$$

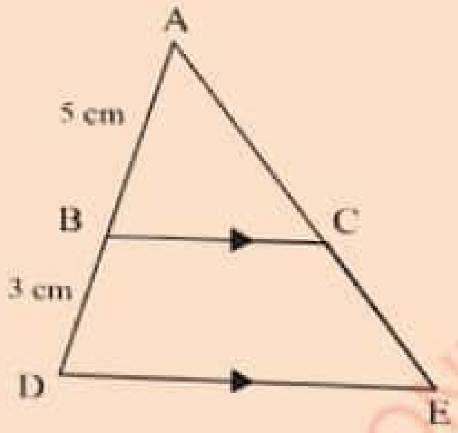
$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta STU} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Therefore, area of $\Delta ABC = 6 \times \frac{9}{4}\text{ cm}^2 = 13.5\text{ cm}^2$.

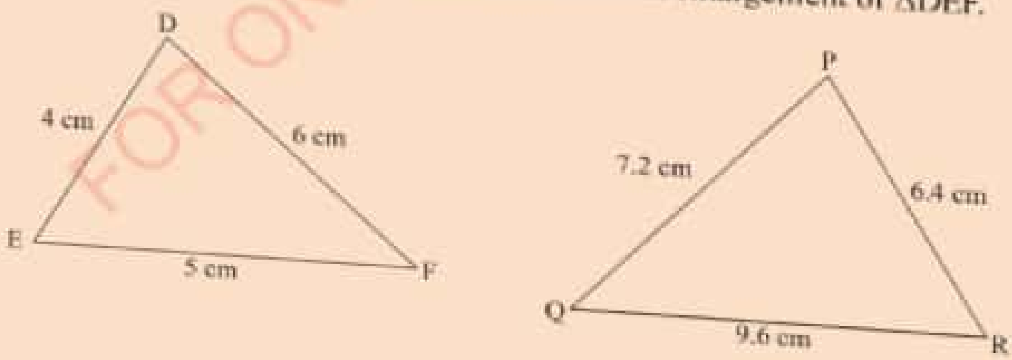
Exercise 7.5

Answer the following questions:

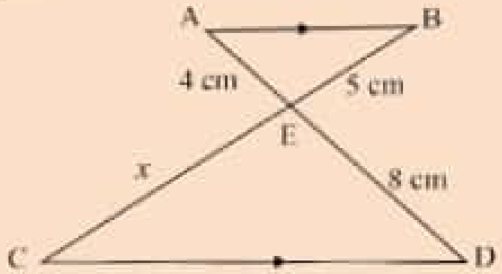
1. In the following figure, $\overline{BC}//\overline{DE}$, $\overline{AB} = 5$ cm, $\overline{BD} = 3$ cm.
   (a) State which triangle is an enlargement of ΔABC.
   (b) Calculate the scale factor of the enlargement.



2. Two triangles are similar but not congruent. Can one triangle be the enlargement of the other?
3. The length of a rectangle is twice the length of another rectangle. Is one rectangle necessarily be an enlargement of the other? Explain your answer.
4. In the following figure, show that ΔPQR is not an enlargement of ΔDEF.



5. Use the following figure to find the value of x and the scale of enlargement of the two triangles.



6. Two windows are similar with a scale factor 3. If the area of the smaller window is 2 cm^2, find the area of the larger window.

7. Triangle XYZ is similar to triangle ABC and $\overline{XY}$ = 8 cm. If the area of the triangle XYZ is 24 cm^2 and the area of the triangle ABC is 96 cm^2 calculate the length of $\overline{AB}$.

Combined transformations

Combined transformation means that two or more transformations are performed on an object. For instance, you could perform a reflection and then a translation on the same object.

Example 7.24

A triangle with vertices A(0, 2), B(1, 0), C(2, 1) is first reflected in the line $x = -1$. It is then enlarged about point (1, −2) with a scale factor 2. Find the vertices of its final image.

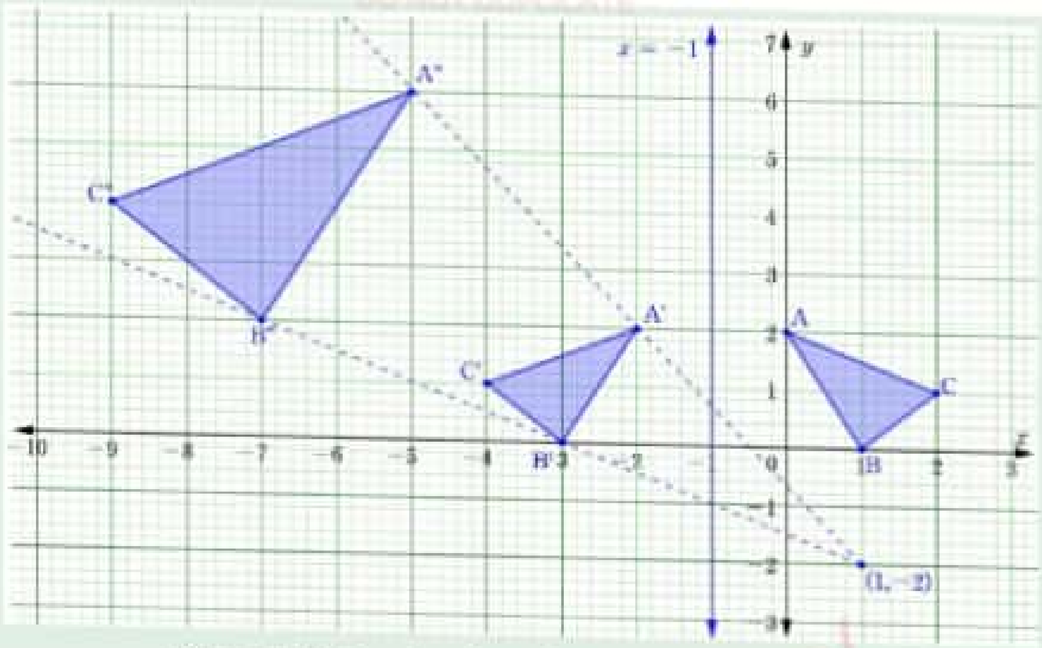


Figure 7.29: *Combined translation of an object ABC*

From the graph in Figure 7.35, the vertices of the final image are A''(–5, 6) B''(–7, 2) and C'(–9, 4).

Exercise 7.6

Answer the following questions:

1. Find the final image in each of the following combined transformations:
   (a) Point A (4, 2) reflected on line $y = x$ and then reflected in the x – axis.
   (b) Point B (–3, 5) translated by T (3, –4) and then rotated through +90° about O(0, 0).
   (c) Point C (4, 4) rotated through +180° about the origin and then reflected on the line $y = x$.

2. Find the image of ΔPQR with vertices P (3, 1), Q (4, 4) and R (2, 4) under:
   (a) Reflection in the y – axis and then rotation through +90° about the origin.
   (b) Reflection in the x – axis and then rotation through –90° about the origin.

3. Triangle ABC with vertices A (1, 0), B (4, –2) and C (3, 2) is enlarged by a scale factor 2 and then reflected in the y – axis. Sketch the object and images on the same xy – plane.

Math F2 – BOOK (DUMMY) Sept 4.indb 253

Chapter summary

1. Reflection is a transformation which reflects all points of a plane figure in a line called the mirror line.
2. Rotation is a transformation which turns all points of an object about a fixed point known as the centre of rotation through a given angle in a clockwise or anticlockwise direction.
3. Translation is a transformation which moves all points of a plane figure in the same direction. It is the result of sliding a figure.
4. Enlargement/dilation is a transformation in which a figure is made larger (magnified) or made smaller (diminished).
5. Scale is a ratio between the measurements of a drawing and the actual measurements.
6. Combined transformation means that two or more transformations are performed on one object. For example, reflection and rotation, reflection and enlargement and so on.

Revision exercise 7

Answer the following questions:

1. List three examples of transformations.
2. Is enlargement a transformation?
3. Find the image of the point B(–3, 6) after a reflection in the x – axis.
4. Find the image of the point D(2, –5) after a reflection in the y – axis.
5. Find the image of the point Q(6, –8) after a rotation through 90° about the origin in the anti – clockwise direction.
6. Find the image of the point N(–2, 3) after a rotation through 270° clockwise about the origin.
7. Find the image of the point (6, 2) after a rotation of 180° about the origin in the anticlockwise direction.

8. A translation takes the origin to (–6, 1), find where it takes (2, –5).

9. A translation takes the points (0, 6) to (0, 10). Where does it take (2, –3)?

10. Draw a parallelogram ABCD with vertices A(2, 5), B(5, 5), C(6, 8) and D(3, 8). Draw the image of the parallelogram ABCD formed by reflection in the y – axis.

11. If $\Delta PQR \sim \Delta LMN$ such that $M\hat{L}N = 40^\circ$ and $Q\hat{R}P = 60^\circ$, find:

 (a) $P\hat{Q}R$ (b) $Q\hat{R}P + R\hat{Q}P$

12. On the map of a certain place, a line segment joining two points is 5 cm long. The distance between the actual points represented is 125 km, find the scale of the map.

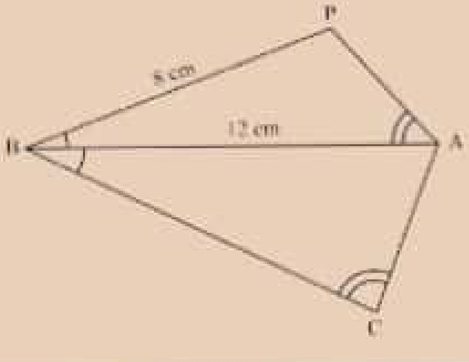
13. The length and width of a rectangular field on a map of a certain town are 10 cm and 8cm, respectively. If the scale of the map is 1:400, find:

 (a) the actual length of the field.
 (b) the actual width of the field.
 (c) the actual area of the field in square metres.

14. If two polygons are similar and the ratio of their corresponding sides is 5:3, what is the enlargement factor?

15. A ladder 5 metres long leans against a vertical wall. The point at which the ladder meets the ground is 3 metres from the wall. Use a scale drawing to find how high up the wall the ladder touches the wall.

16. Use the following figure to find the enlargement factor of Δ ABC and the scale factor of its area, and hence find its area given that the area of Δ PBA is 10 cm².



Pythagoras' theorem

Introduction

The Pythagoras' theorem is a fundamental theorem which relates the three sides of a right – angled triangle . The theorem bears the name of a Greek Mathematician, Pythagoras, who was born on the Island of Samos in Greece. In this chapter, you will learn about the Pythagoras' theorem, its proof and applications. It is applicable in architecture and engineering science to calculate slopes of buildings and calculating the distance between two points using a reference point, or the magnitude of a vector. The competences developed in this chapter will enable you to obtain the lengths or heights of objects without physically taking measurements.

Pythagoras' theorem

Consider the right – angled triangle in Figure 8.1:

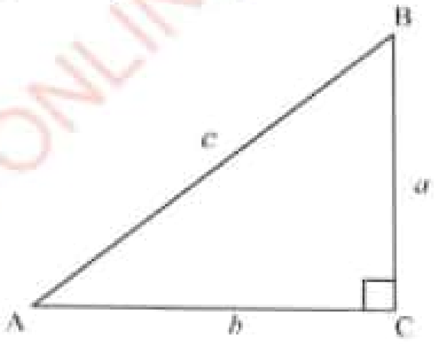


Figure 8.1 *A right – angled triangle*

The longest side, which is opposite to the right angle, is called the **hypotenuse**, and the other two sides which form the right angle are called the **adjacent** sides. Pythagoras' theorem states that: *"In a right – angled triangle the square of the hypotenuse is equal to the sum of the squares of the adjacent sides."*

Therefore, if two sides of a right – angled triangle are known, the third side can be determined by using Pythagoras' theorem. From Figure 8.1, the Pythagoras' theorem is written as:

$$\left(\overline{BC}\right)^2 + \left(\overline{AC}\right)^2 = \left(\overline{AB}\right)^2$$

That is, $a^2 + b^2 = c^2$.

Activity 8.1 Establishing a relationship between the sides of a right – angled triangle.

Materials required: Manila paper, ruler, pencil

Steps

1. Take a sheet of manila paper and draw five right – angled triangles of different sizes.
2. Measure the lengths of the sides of the triangles.
3. Square the length of each side.
4. For each triangle, compare the squared length of the longest side with the sum of squared lengths of the adjacent sides.
5. State the relationship you observed.

Consider Figure 8.2 which represents triangle ABC with $A\hat{C}B = 90°$.

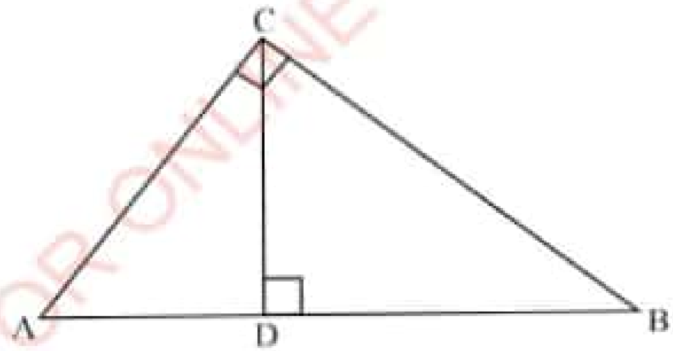


Figure 8.2: *Triangle* ABC

Required to prove: $\left(\overline{AC}\right)^2 + \left(\overline{BC}\right)^2 = \left(\overline{AB}\right)^2$

Construction: Through C, draw $\overline{CD}$ perpendicular to $\overline{AB}$.

Proof:

$A\hat{D}C = A\hat{C}B$ (each measures 90^0)

$C\hat{A}B = C\hat{A}D$ (Common)

$A\hat{C}D = A\hat{B}C$ (third angle of triangles)

$\Delta ABC \sim \Delta ACD$ (AA – similarity theorem)

$$\therefore \frac{\overline{BA}}{\overline{AC}} = \frac{\overline{BC}}{\overline{CD}} = \frac{\overline{AC}}{\overline{AD}}$$

In particular, $\frac{\overline{AC}}{\overline{AD}} = \frac{\overline{BA}}{\overline{AC}}$

$\therefore \left(\overline{AC}\right)^2 = \overline{AD} \times \overline{AB}$. (1)

In the same way:

$\Delta ABC \sim \Delta CBD$

$$\therefore \frac{\overline{AB}}{\overline{CB}} = \frac{\overline{BC}}{\overline{BD}} = \frac{\overline{AC}}{\overline{CD}}$$

In particular, $\frac{\overline{AB}}{\overline{CB}} = \frac{\overline{BC}}{\overline{BD}}$

$\left(\overline{BC}\right)^2 = \overline{BD} \times \overline{AB}$ (2)

Adding (i) and (ii) results into,

$$\begin{aligned}\left(\overline{AC}\right)^2 + \left(\overline{BC}\right)^2 &= \left(\overline{AD} \times \overline{AB}\right) + \left(\overline{BD} \times \overline{AB}\right)\\ &= \overline{AB}\left(\overline{AD} + \overline{BD}\right)\\ &= \overline{AB} \times \overline{AB} \quad \left(\text{since } \overline{AD} + \overline{BD} = \overline{AB}\right)\\ &= \left(\overline{AB}\right)^2\end{aligned}$$

Therefore, $\left(\overline{AC}\right)^2 + \left(\overline{BC}\right)^2 = \left(\overline{AB}\right)^2$.

Alternatively;

Consider the square PQRS with side length $a + b$ as shown in Figure 8.3.

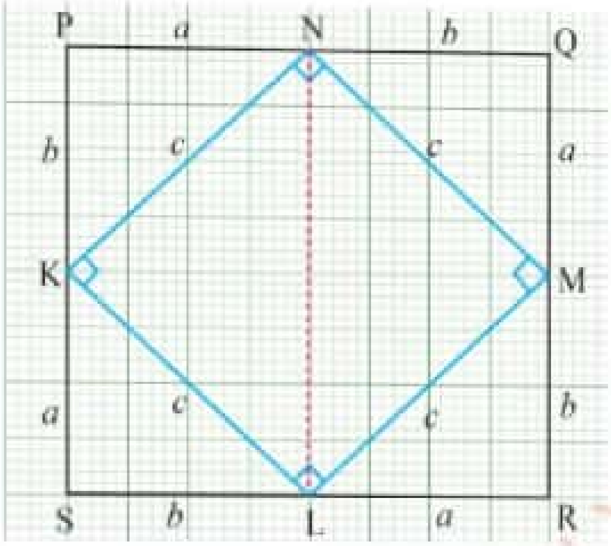


Figure 8.3: *Rectangle PQRS.*

Area of PQRS = Area of NMLK + Area of four equal triangles

$$(a+b)(a+b) = c \times c + 4\left(\frac{1}{2} \times a \times b\right)$$

$$a^2 + ab + ab + b^2 = c^2 + 4 \times \frac{1}{2} \times a \times b$$

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

$$a^2 + b^2 = c^2 + 2ab - 2ab$$

$$a^2 + b^2 = c^2$$

Hence, the square of the hypotenuse is equal to the sum of the squares of the adjacent sides. The converse of the theorem is: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the angle between the two sides is a right angle.

If $a^2 + b^2 \neq c^2$ then the triangle is not right – angled.

Example 8.1

The adjacent sides of a right – angled triangle have lengths 5 cm and 12 cm. Find the length of the hypotenuse.

Solution

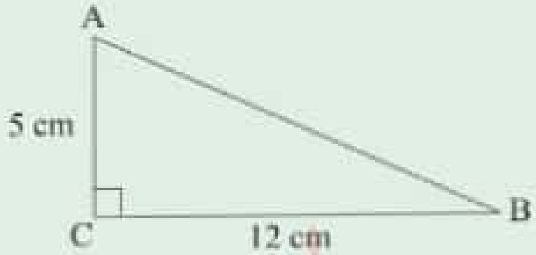
Construct ΔABC with $A\hat{C}B = 90^\circ$, $\overline{AC} = 5\text{cm}$ and $\overline{BC} = 12$ cm. Required to find $\overline{AB}$.

Using Pythagoras' Theorem

$$(\overline{AB})^2 = (\overline{AC})^2 + (\overline{BC})^2 = 5^2 + 12^2$$

$$= 25 + 144 = 169$$

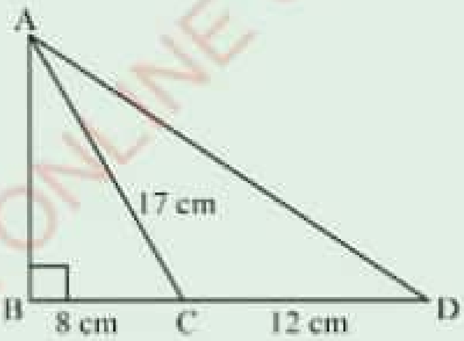
$$\overline{AB} = \sqrt{169} = 13 \text{ cm}$$



Therefore, the length of the hypotenuse is 13 cm.

Example 8.2

In the following figure, $\overline{AC} = 17$ cm, $\overline{BC} = 8$ cm, and $\overline{CD} = 12$ cm, find $\overline{AD}$.



Solution

Applying Pythagoras' theorem:

In ΔABC,

$$(\overline{AB})^2 + (\overline{BC})^2 = (\overline{AC})^2$$

$(\overline{AB})^2 + 8^2 = 17^2$

$(\overline{AB})^2 = 289 - 64 = 225$

$\overline{AB} = \sqrt{225} = 15$

In ΔABD,

$$(\overline{AD})^2 = (\overline{AB})^2 + (\overline{BD})^2$$
$$= (\overline{AB})^2 + (\overline{BC} + \overline{CD})^2$$
$$= 15^2 + (8+12)^2 = 225 + 20^2$$
$$= 225 + 400$$

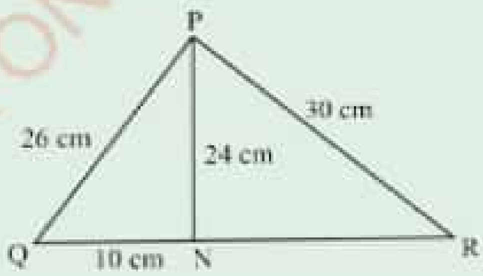
$(\overline{AD})^2 = 625$

$\overline{AD} = \sqrt{625} = 25$

Therefore, $\overline{AD} = 25$ cm.

Example 8.3

In the following figure, prove that $P\hat{N}R = 90°$, and hence calculate $\overline{NR}$.



Solution

From ΔQNP,

$$(\overline{QN})^2 + (\overline{PN})^2 = 10^2 + 24^2$$
$$= 100 + 576$$
$$= 676$$
$$(\overline{QP}) = 26^2 = 676$$

Since $(\overline{QP})^2 = (\overline{QN})^2 + (\overline{PN})^2 = 676$

Then, $Q\hat{N}P = 90°$ (Pythagoras' theorem)

Hence, $P\hat{N}R = 90°$ (supplement of $Q\hat{N}P$)

Applying Pythagoras' theorem to ΔPNR:

$$(\overline{PN})^2 + (\overline{NR})^2 = (\overline{PR})^2$$
$$24^2 + (\overline{NR})^2 = 30^2$$
$$(\overline{NR})^2 = 900 - 576 = 324$$
$$(\overline{NR}) = \sqrt{324}$$

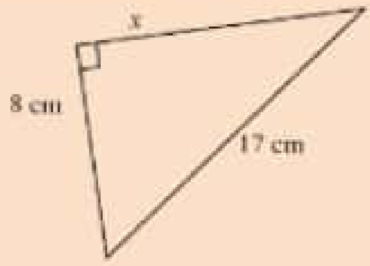
Therefore, $\overline{NR} = 18$ cm.

Exercise 8.1

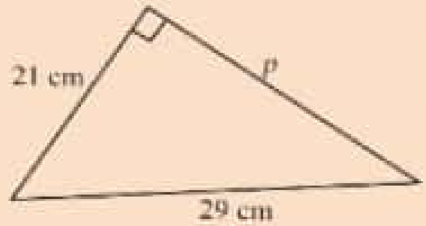
Answer the following questions:

1. Calculate the unknown side in each of the following triangles:

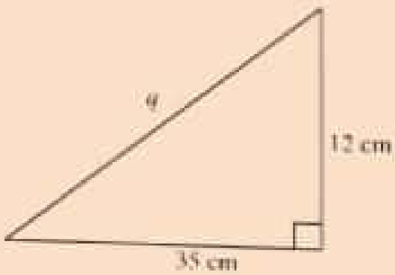
(a)



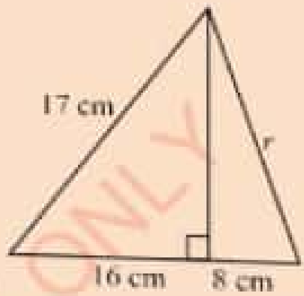
(b)



(c)



(d)



2. Given triangle ABC, where $\hat{B} = 90°$, find the lengths of the sides which are not given:

   (a) $\overline{AC} = 26$ cm, $\overline{AB} = 10$ cm

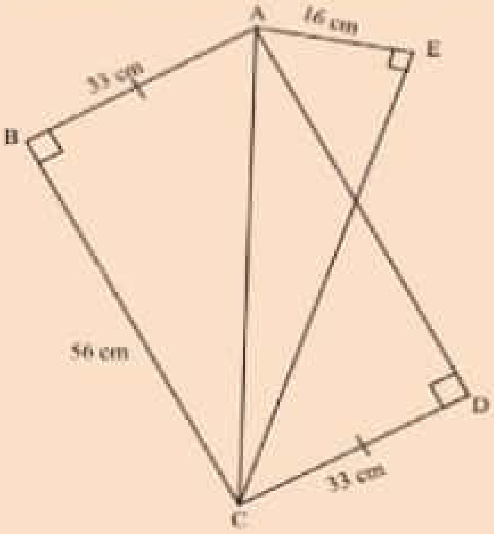
   (b) $\overline{AB} = 20$ cm, $\overline{BC} = 21$ cm

   (c) $\overline{AC} = 25$ cm, $\overline{AB} = 15$ cm

   (d) $\overline{AB} = 28$ cm, $\overline{BC} = 45$ cm

3. Calculate the length of a diagonal of a square with sides 7 cm.
4. A man travels 15 km due north and then 8 km due west. How far is he from his starting point?

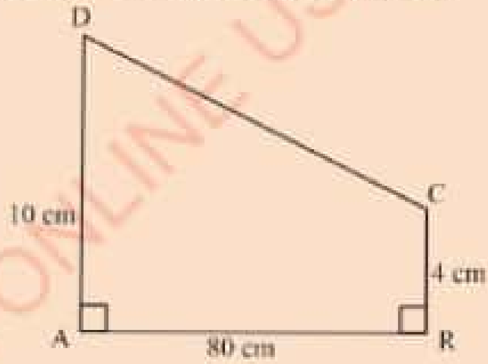
5. In the following figure, calculate the value of $\overline{AC}$, $\overline{AD}$, and $\overline{EC}$.



6. A quadrilateral ABCD is such that, $D\hat{A}B = B\hat{C}D = 90°$. Prove that $\left(\overline{AB}\right)^2 + \left(\overline{AD}\right)^2 = \left(\overline{DB}\right)^2$.

7. The sides of a rectangle are 5 cm and 7 cm. Find the length of a diagonal.

8. Use the following figure to find the length of $\overline{CD}$.



9. Find the sides of a rhombus whose diagonals are 6 cm and 10 cm.

10. A ladder touches the top of a wall 18 m high when it is 2 m away from the wall. Find the length of the ladder.

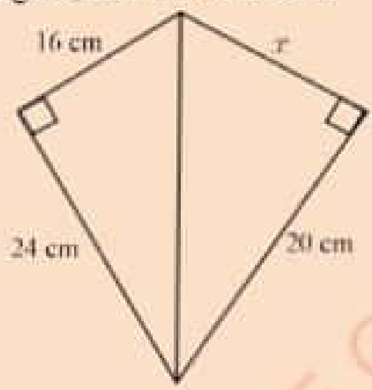
11. In triangle ABC, $\overline{AB} = \overline{AC} = 13$ cm, $\overline{BC} = 10$ cm. Find the length of the perpendicular from A to $\overline{BC}$ and the area of the triangle ABC.

12. Calculate the altitude of an equilateral triangle with sides of 8 cm long each.

13. In a quadrilateral ABCD, $B\hat{A}D = 90°$. If $\left(\overline{AB}\right)^2 + \left(\overline{AD}\right)^2 = \left(\overline{CB}\right)^2 + \left(\overline{CD}\right)^2$, prove that $B\hat{C}D = 90°$.

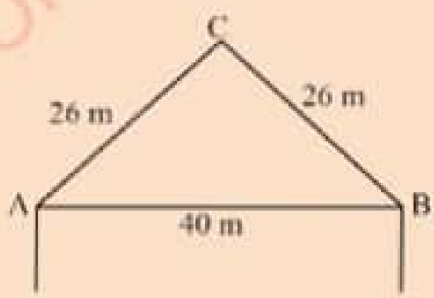
14. In a quadrilateral PQRS, PQR and PRS are right – angled triangles. If $\overline{QR} = 9$ cm, and $\overline{RS} = 8$ cm, find the length of $\overline{PS}$ and the area of PQRS.

15. In the following figure, find the value of x.



16. The length of a diagonal of a square is 72 cm. Find the length of the sides of the square.

17. In the following figure, A, B and C is a roof of a godown with its highest point at C. The horizontal level of the walls AB is 7 m above the ground. How far above the ground is the highest point of the roof ?



18. In a quadrilateral ABCD, $\overline{AB} = 9$ cm, $\overline{BC} = 12$ cm, $\overline{AD} = 25$ cm, $\overline{CD} = 20$ cm and $A\hat{B}C = 90°$. Find the area of ABCD.

19. Determine whether each of the following set of measures can be sides of a right – angled triangle:

(a) 3, 4, 5 (b) 7, 24, 25 (c) 12, 34, 37

(d) 9, 40, 41 (e) 20, 21, 31 (f) 20, 48, 52

20. Consider a right – angled triangle whose adjacent sides are $m^2 - n^2$ and $2mn$. Show that the length of the hypotenuse side is $m^2 + n^2$.

21. The perimeter of an equilateral triangle is 45 cm. Find the height of the triangle.

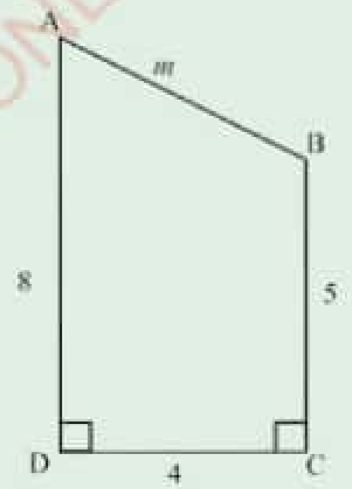
Application of Pythagoras' theorem

If two straight lines meet at a right angle, Pythagoras' theorem allow you to calculate the length of the line connecting the end points of the lines.

This is one example how Pythagoras' theorem is applied in real life situation. Generally, the Pythagoras theorem is applicable in calculating distances between two points as well as heights of buildings and other structures.

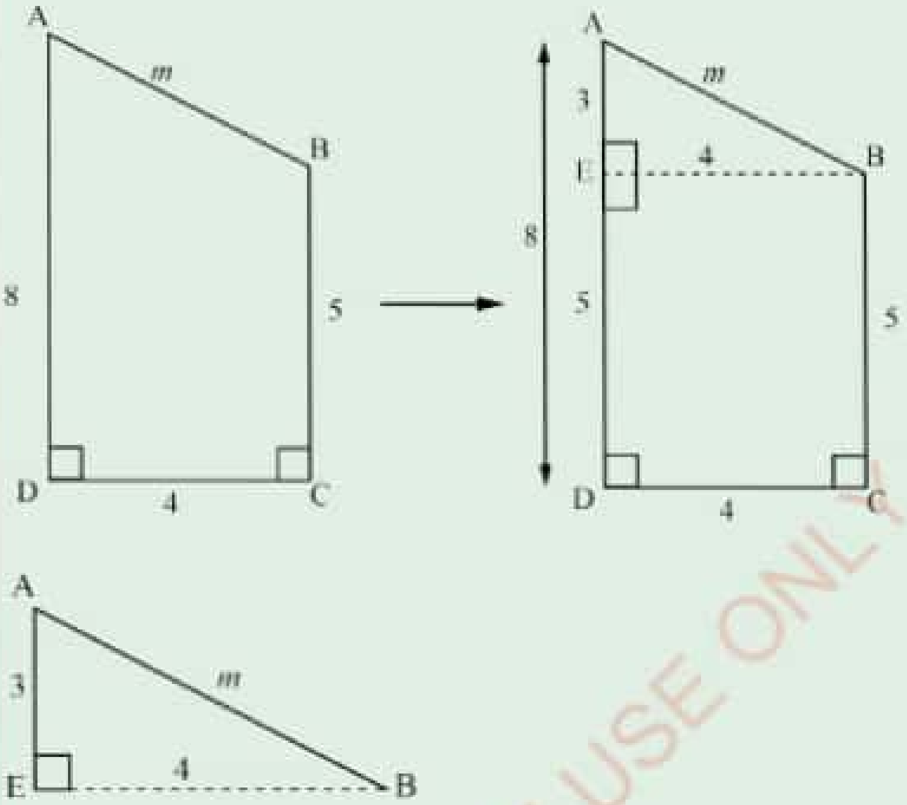
Example 8.4

Given the following figure, find the value of m.



Solution

In order to use Pythagoras' theorem, we need to have a right – angled triangle



Now, in the right – angled triangle ABE,

$m^2 = 3^2 + 4^2$ (Pythagoras' theorem)

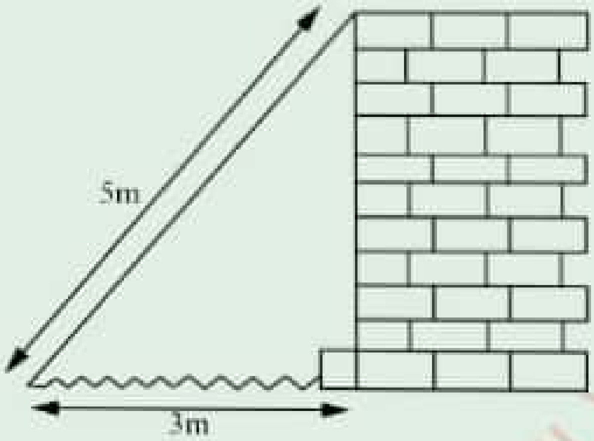
$m^2 = 9 + 16 = 25$

$m = \sqrt{25}$

Therefore, the value of m is 5.

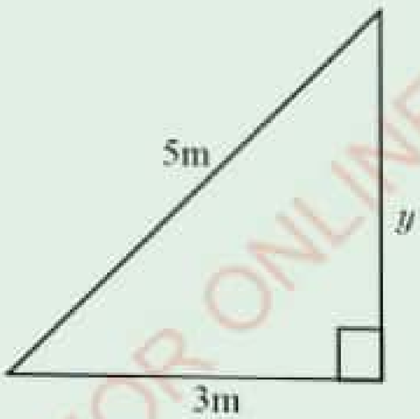
Example 8.5

A ladder 5m long leans against a wall. The foot of the ladder is 3m from the base of the wall. At what height from the ground does the top of the ladder leans against the wall?



Solution

The above diagram can be redrawn to represent a right – angled triangle:



By Pythagoras theorem,

$y^2 + 3^2 = 5^2$

$y^2 + 9 = 25$

$y^2 = 25 - 9$

$y^2 = 16$

$y = \sqrt{16}$

$y = 4$ m

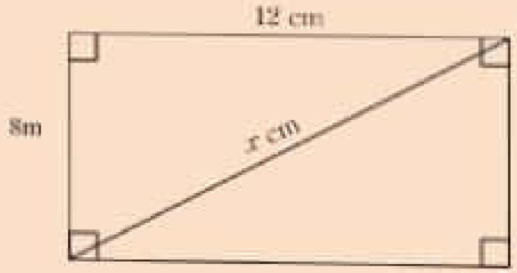
The top of the ladder reaches 4 m up the wall.

Exercise 8.2

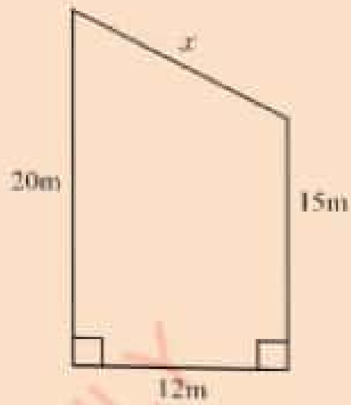
Answer the following questions:

1. Find the unknown length in each of the following figures. Give the answer correct to 3 significant figures.

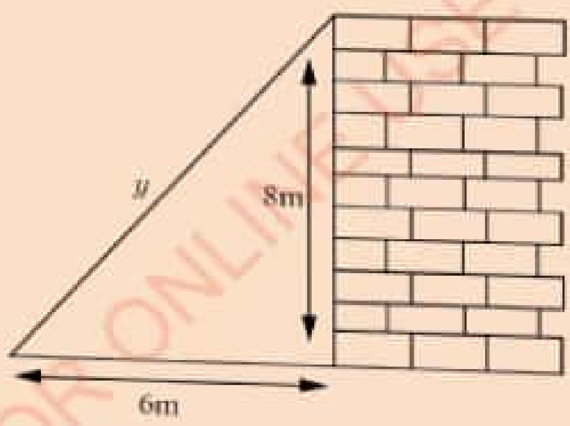
(a)



(b)



2. A ladder leans against a wall. If the ladder reaches 8 m up the wall and its foot is 6 m away from the base of the wall, find the length of the ladder.



3. A woman walks 500 m up the side of a hill. If she has climbed a vertical distance of 300 m, find the horizontal distance through which she has travelled.
4. A rectangular field, of 80 m long, is to be fenced with a wire. If the diagonal of the field is 100 m long, find:
   (a) width of the field.
   (b) length of the wire needed to fence the field.

5. The hypotenuse of an isosceles right – angled triangle is 8 cm. Is there enough information to find the length of the adjacent sides? If so, find their lengths. If not, explain why not.
6. A painter placed a 20 metre ladder against the wall of a house so that the base of the ladder is 4 metre away from the wall. How high does the ladder touch the wall from the ground?

Chapter summary

1. Pythagoras' theorem states that: In any right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of lengths of the adjacent sides.

2. The converse of the Pythagoras' theorem states that: If the square of one side of a triangle is not equal to the sum of the squares of the other two sides, then the triangle is not right – angled.

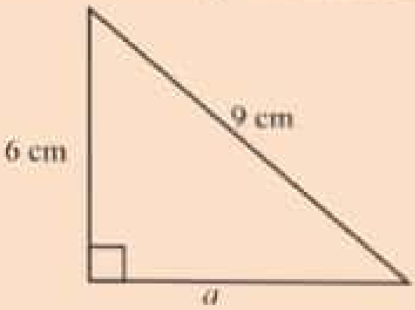
3. Pythagoras' theorem has applications in architectural works such as in construction of buildings, bridges and walls. It is also used to find the width of rivers and heights of buildings.

Revision exercise 8

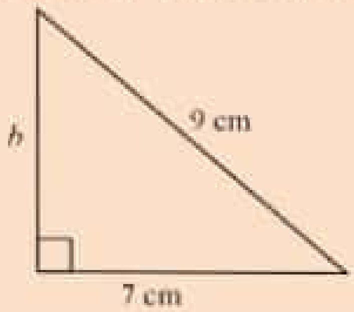
Answer the following questions:

1. Find the unknown length in each of the following right – angled triangles:

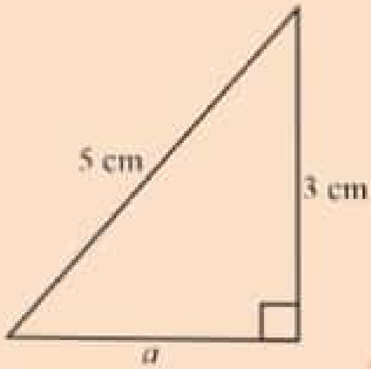
(a)



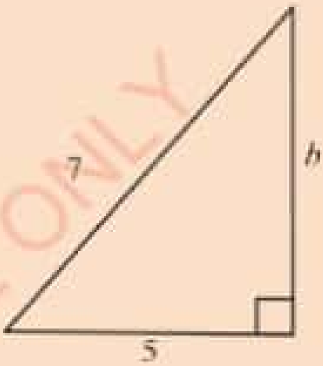
(b)



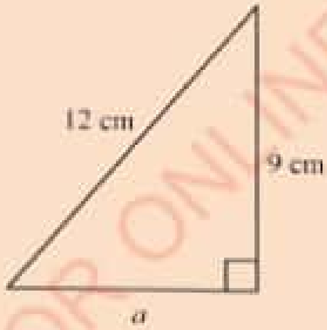
(c)



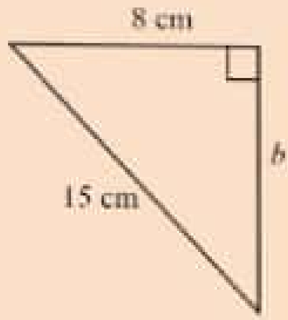
(d)



(e)

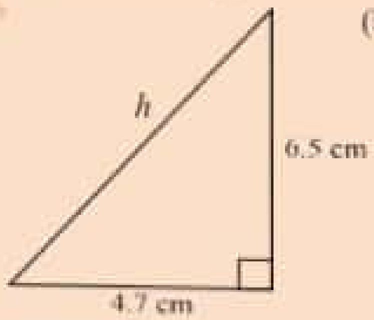


(f)

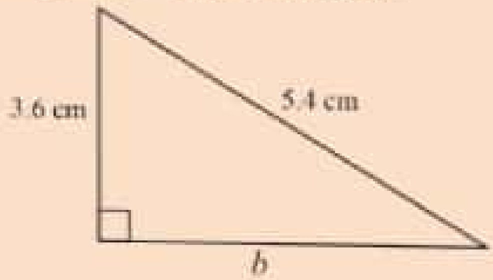


2. Find the unknown length in the following right – angled triangles.

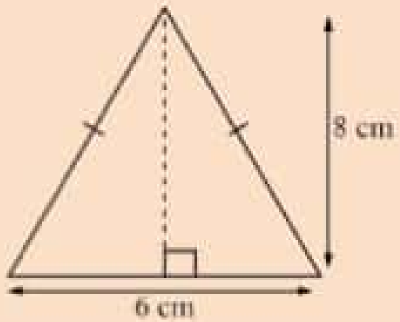
(a)



(b)

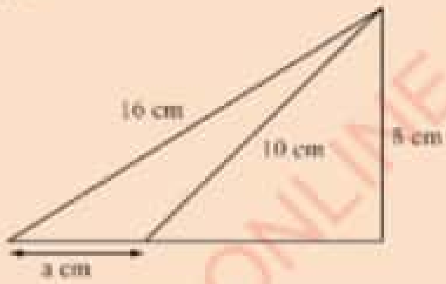


3. A rectangle has the dimensions 15 cm by 11 cm. What is the length of its diagonal?
4. The following figure is an isosceles triangle of height 8 cm and base 6 cm. What is the length of each side?

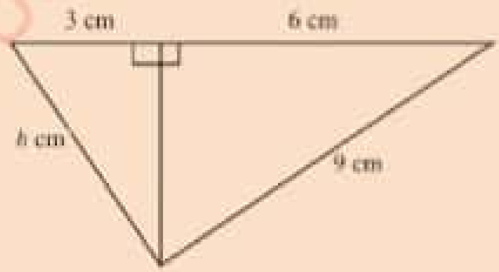


5. Find the length of the diagonal of a square whose sides are 18 cm.
6. The diagonal of a square is 25 cm long. What is the length of its sides?
7. Calculate the length of the sides marked with letters in the following figures:

(a)



(b)



8. A ladder 10 m long leans against a wall. The foot of the ladder is 6 m from the base of the wall. How high does the top of the ladder reach?
9. A rope of length 39 m is tied to the top of a flag pole. The other end of the rope is fixed at a point 36 m from the base of the flagpole. How high is the flag pole?

Chapter Nine

Trigonometry

Introduction

The relationship between sides and angles of triangles is one of the important subject areas in mathematics. Trigonometry is a branch of mathematics that deals with the relationship between lengths of the sides and angles of triangles. In this chapter you will learn about trigonometric ratios, trigonometric ratios of special angles, trigonometric tables as well as calculations which involve angles of elevation and depression. Trigonometry is used in navigation to find the distance of the shore from a point in the sea. It is also used in calculating the height of tides in oceans. The competencies developed in this chapter will enable you to find heights of buildings, trees, hills, or mountains, width of a river, slopes of roofs, and many other objects without physically measuring them but only using angle measuring devices.

Trigonometric ratios

Sine, Cosine, and Tangent are the basic trigonometric ratios abbreviated as 'sin', 'cos', and 'tan'. These are referred to as ratios because they are defined by ratios of the sides of a right – angled triangle. Other trigonometric ratios derived from these basic ratios are secant, cosecant and cotangent. These are reciprocals of the three basic trigonometric ratios. The sine, cosine and tangent ratios are defined in terms of the sides and angles of a right – angled triangle.

Activity 9.1 Identifying the properties of a right – angled triangle.

Using a protractor and ruler:

1. Draw three right – angled triangles with different sizes but the corresponding interior angles are congruent (that is, ΔACB, ΔAED and ΔAGF are congruent) and name $C\hat{A}B = E\hat{A}D = G\hat{A}F$.
   (a) Measure the lengths $\overline{AB}$ and $\overline{AC}$.

(b) Compute the ratio $\frac{\overline{AB}}{\overline{AC}}$.

(c) Measure the length of $\overline{AD}$ and $\overline{AE}$.

(d) Compute the ratio $\frac{\overline{AB}}{\overline{AE}}$.

(e) Measure the lengths $\overline{AF}$ and $\overline{AG}$.

(f) Compute the ratio $\frac{\overline{AF}}{\overline{AG}}$.

(g) What do you notice about your answers from (b), (d) and (f)?

2. What is the basic condition for a triangle to satisfy the Pythagoras theorem?
3. Draw two similar right-angled triangles in which the ratio of their corresponding sides is $\frac{2}{3}$.
4. What are the measures of the angles in each triangle? What are the lengths of the sides of each triangle?

Consider the right – angled triangle ABC shown in Figure 9.1.

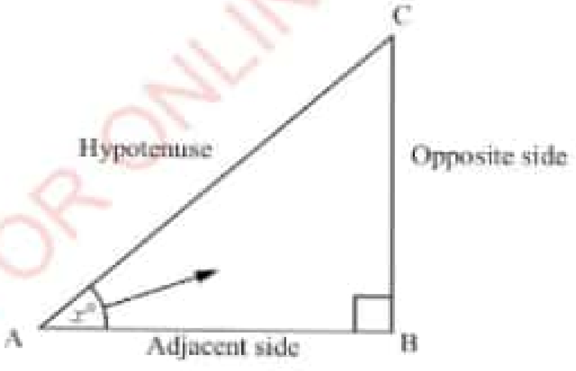


Figure 9.1: *Right – angled triangle ABC*

In relation to angle $C\hat{A}B = x^\circ$, side $\overline{AC}$ is called the **hypotenuse**, side $\overline{AB}$ is called the **adjacent** side, and side $\overline{BC}$ (opposite to the angle x°) is called the **opposite** side.

Consider the similar triangles shown in Figure 9.2.

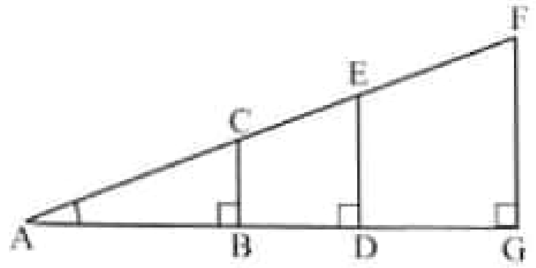


Figure 9.2 *Similar triangles*

$$\frac{\overline{CB}}{\overline{AB}} = \frac{\overline{ED}}{\overline{AD}} = \frac{\overline{FG}}{\overline{AG}} = t,$$

where t is a constant ratio. This constant ratio is called the **tangent** of the angle at vertex A and is written in short as **tan $\hat{A}$.**

Similarly, $\frac{\overline{BC}}{\overline{AC}} = \frac{\overline{DE}}{\overline{AE}} = \frac{\overline{GF}}{\overline{AF}} = s,$

where s is a constant ratio. This constant ratio s is called the **sine** of the angle at vertex A and is written in short as **sin $\hat{A}$.**

Likewise, $\frac{\overline{BA}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AE}} = \frac{\overline{AG}}{\overline{AF}} = c,$

where c is a constant ratio. This constant ratio c is called the **cosine** of the angle at vertex A and is written in short as **cos $\hat{A}$.**

Consider Figure 9.3 to obtain the trigonometrical ratios of an angle.

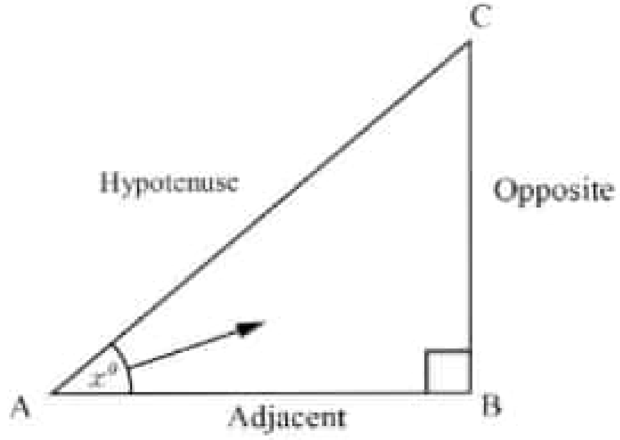


Figure 9.3: *Trigonometric ratios of angle* $\hat{A}$

The length of the sides of a right – angled triangle in Figure 9.3 are used to define the basis trigonometric ratios as follows:

$$\tan \hat{A} = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\sin \hat{A} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos \hat{A} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

The following is a useful way of remembering these definitions:

SO	TO	CA
H	A	H

where $S = \frac{O}{H}$ is the definition of $\sin \hat{A}$

$T = \frac{O}{A}$ is the definition of $\tan \hat{A}$

$C = \frac{A}{H}$ is the definition of $\cos \hat{A}$

Example 9.1

In a right-angled triangle ABC, $\overline{AB} = 3$ cm, $\overline{BC} = 4$ cm and $\overline{AC} = 5$ cm.

Find the value of the following:

(a) $\sin \hat{A}$ (b) $\cos \hat{A}$ (c) $\tan \hat{A}$

Solution

ABC is a right – angled triangle such that,

$A\hat{B}C = 90°$. It follows that

(a) $\sin \hat{A} = \frac{\text{opposite side}}{\text{hypotenuse}}$

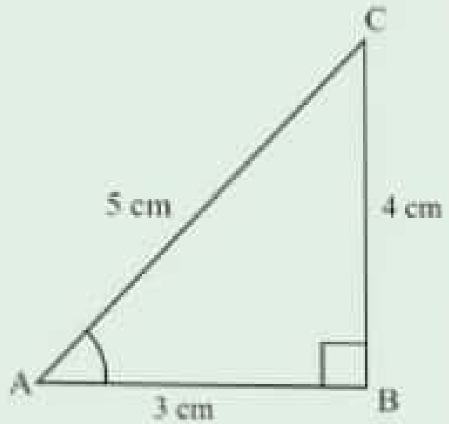
$= \frac{\overline{BC}}{\overline{AC}} = \frac{4 \text{ cm}}{5 \text{ cm}}$

Therefore, $\sin \hat{A} = \frac{4}{5}$.

(b) $\cos \hat{A} = \frac{\text{adjacent side}}{\text{hypotenuse}}$

$= \frac{\overline{AB}}{\overline{AC}} = \frac{3 \text{ cm}}{5 \text{ cm}}$

Therefore, $\cos \hat{A} = \frac{3}{5}$.



(c) $\tan \hat{A} = \frac{\text{opposite side}}{\text{adjacent side}}$

$= \frac{\overline{BC}}{\overline{AB}} = \frac{4 \text{ cm}}{5 \text{ cm}}$

Therefore, $\tan \hat{A} = \frac{4}{3}$.

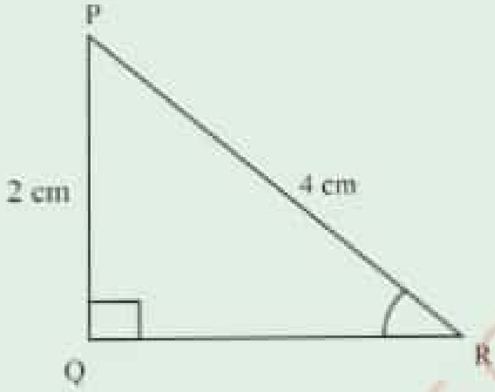
Math F2 – (BOOK) (DUMMY) Sept 4.indb 277

Example 9.2

In triangle PQR, angle PQR is a right-angle. If $\overline{RP} = 4$ cm and $\overline{PQ} = 2$ cm; find

(a) $\overline{QR}$ (b) $\tan \hat{R}$ (c) $\cos \hat{R}$

Solution



(a) $\overline{QR}^2 + 2^2 = 4^2$

$\overline{QR}^2 = 16 - 4 = 12$

$\overline{QR} = \sqrt{12}$

Therefore, $\overline{QR} = 2\sqrt{3}$ cm.

(b) $\tan \hat{R} = \dfrac{\text{opposite side}}{\text{adjacent side}}$

$= \dfrac{\overline{PQ}}{\overline{QR}}$

$= \dfrac{2}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

Therefore, $\tan \hat{R} = \dfrac{\sqrt{3}}{3}$.

(c) $\cos \hat{R} = \dfrac{\text{adjacent side}}{\text{hypotenuse side}}$

$= \dfrac{\overline{QR}}{\overline{PR}} = \dfrac{2\sqrt{3}}{4}$

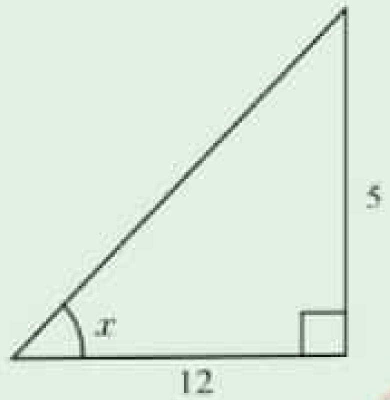
Therefore, $\cos \hat{R} = \dfrac{\sqrt{3}}{2}$.

Example 9.3

If $\tan x = \frac{5}{12}$, find the value of $\cos x$.

Solution

By definition, $\tan x = \frac{5}{12}$ may be represented as shown in the following triangle.



Using Pythagoras' theorem, the third side (hypotenuse) is

$$12^2 + 5^2 = \sqrt{144+25} = \sqrt{169} = 13$$

$$\cos x = \frac{\text{adjacent side}}{\text{hypotenuse side}} = \frac{12}{13}$$

Therefore, $\cos x = \frac{12}{13}$.

Example 9.4

If XYZ is a triangle such that $X\hat{Y}Z = 90°$, $\overline{ZY} = 5$ cm, $Z\hat{X}Y = 30°$ and $\overline{ZX} = 10$ cm. Find the value of $\sin 30°$.

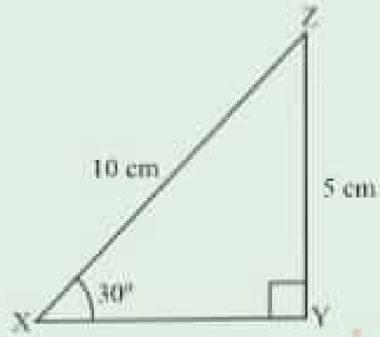
Solution

$$\text{Sin } 30° = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$= \frac{5}{10}$$

$$= 0.5$$

Therefore, $\sin 30° = 0.5$



Exercise 9.1

Answer the following questions:

1. In triangle LMN, $L\hat{N}M = 90°$, $\overline{LM} = 10$ cm, $\overline{MN} = 6$ cm and $\overline{LN} = 8$ cm. Find:

   (a) $\tan \hat{M}$ (b) $\sin \hat{M}$ (c) $\cos \hat{M}$

2. In a triangular plot ABC, $B\hat{A}C = 90°$, $\overline{AB} = 8$ m, $\overline{AC} = 15$ m and $\overline{BC} = 17$ m. Find the values of each of the following:

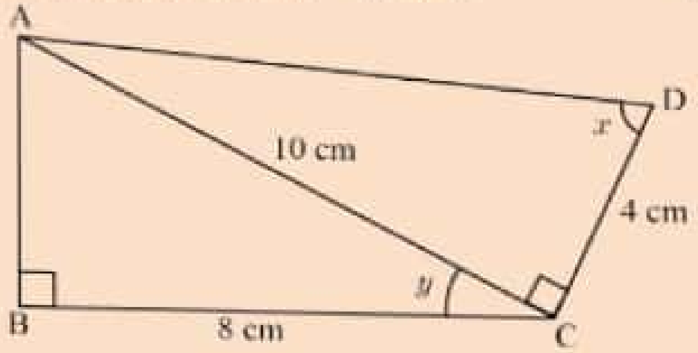
   (a) $\sin \hat{C}$ (b) $\tan \hat{C}$ (c) $\cos \hat{C}$

3. In a triangle RST, $R\hat{S}T = 90°$, $\overline{RS} = 4$ cm and $\overline{TS} = 3$ cm. Find:

   (a) $\overline{TR}$ (b) $\cos \hat{R}$ (c) $\sin \hat{R}$

4. A rectangular field is 100 m long and 50 m wide. If one of its diagonals makes an angle x with the length, find the value of $\tan x$.

5. Use the following figure to find: (a) $\tan x$ (b) $\sin y$



6. If $\sin x = \frac{4}{5}$, find the value of (a) $\tan x$ (b) $\cos x$

7. If $\cos x = \frac{15}{17}$, find the value of:
   (a) $\sin x$ (b) $\tan x$

8. Given that $\sin 30^\circ = \frac{1}{2}$, find the value of $\cos 30^\circ$.

9. In a triangle ABC, $\overline{AB} = c$, $\overline{BC} = a$, $\overline{AC} = b$ and $A\hat{B}C = 90^\circ$. Find in terms of a, b or c:
   (a) $\sin \hat{C}$ (b) $\cos \hat{C}$
   (c) $\tan \hat{C}$ (d) $\frac{\sin \hat{C}}{\cos \hat{C}}$
   (e) What is the relationship between $\tan \hat{C}$ and $\frac{\sin \hat{C}}{\cos \hat{C}}$?

10. In a right-angled triangle, the length of the hypotenuse is 10 cm and one of its angles is 70°. Calculate the length of the shortest side (use $\cos 70^\circ = 0.342$).

Trigonometric ratios of special angles

Special angles are those angles whose values of their trigonometric ratios can be found by using simple ratios. The special angles are 30°, 45°, and 60°.

Consider an equilateral triangle ABC shown in Figure 9.4. Let the length of each side be 2 units. The altitude from A bisects $\overline{BC}$ at D.

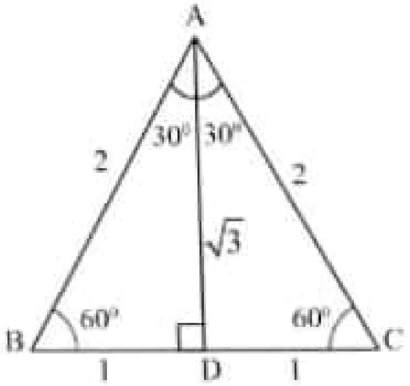


Figure 9.4: *A triangle with special angles 30° and 60°*

$\overline{BD} = \overline{DC} = 1$(equal halves of side $\overline{BC}$).

$\overline{BD}^2 + \overline{AD}^2 = 2^2$ (Pythagoras' theorem)

$1^2 + \overline{AD}^2 = 2^2$

$\overline{AD}^2 = 4 - 1 = 3$

$\therefore \quad \overline{AD} = \sqrt{3}$

$$\sin 60^\circ = \frac{\overline{AD}}{\overline{AB}} = \frac{\sqrt{3}}{2} \qquad \sin 30^\circ = \frac{\overline{BD}}{\overline{AB}} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{\overline{BD}}{\overline{AD}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \tan 30^\circ = \frac{\overline{BD}}{\overline{AD}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cos 60^\circ = \frac{\overline{BD}}{\overline{AB}} = \frac{1}{2} \qquad \cos 30^\circ = \frac{\overline{AD}}{\overline{AB}} = \frac{\sqrt{3}}{2}.$$

Consider the isosceles triangle PQR in Figure 9.5 in which the base angles is 45° and $\overline{PQ} = \overline{RQ} = 1$.

$\overline{PR} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan 45° = \frac{\overline{PQ}}{\overline{RQ}} = \frac{1}{1} = 1$

$\sin 45° = \frac{\overline{QR}}{\overline{PR}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos 45° = \frac{\overline{PQ}}{\overline{PR}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

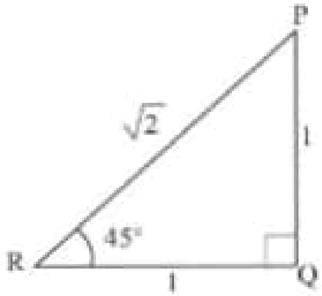


Figure 9.5: *Isosceles triangle with special angle 45°*

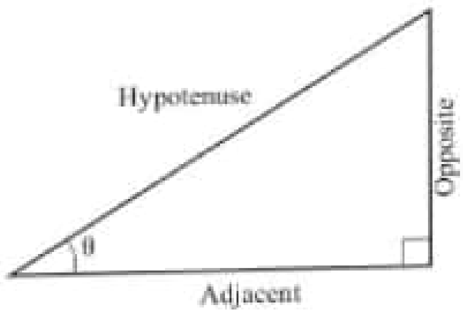
These results are summarized in Table 9.1.

Table 9.1: Trigonometric ratios for the special angles.

x	$\sin x$	$\cos x$	$\tan x$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Values of sin 90°, cos 90° and tan 90°

As θ goes to 90°
Hypotenuse becomes very long
Opposite side becomes very long
Adjacent side remains the same



Thus, $\sin 90° = \frac{\text{very long}}{\text{very long}}$ approaches 1

$\cos 90° = \frac{\text{adjacent}}{\text{very long}}$ approaches 0

$\tan 90° = \frac{\text{very long}}{\text{adjacent}}$ becomes very large (undefined).

Values for sin 0°, cos 0° and tan 0°

Consider the following right – angled triangle in Figure 9.6.

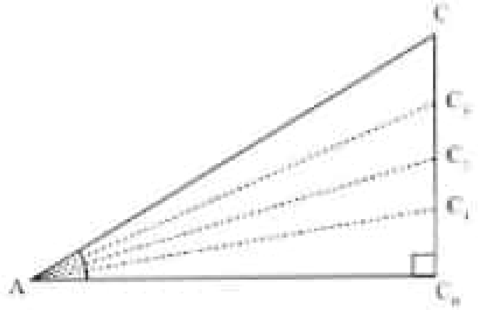


Figure 9.6: *Movement of* $\overline{AC}$ towards $\overline{AC}_0$.

When $\overline{AC}$ collapses to $\overline{AC}_0$, angle $C\hat{A}C_0$ become 0 then, the hypotenuse and adjacent sides overlap. This means that the opposite side becomes zero. Thus,

$$\sin 0° = \frac{\text{opposite side}}{\text{hypotenuse side}} = \frac{0}{\text{hypotenuse}} = 0$$

Therefore, **sin 0° = 0**

$$\cos 0° = \frac{\text{adjacent}}{\text{hypotenuse}} = 1$$

(since the hypotenuse overlaps with the adjacent side and they become equal)

Therefore, **cos 0° = 1**

$$\tan 0° = \frac{\text{opposite}}{\text{adjacent}} = \frac{0}{\text{adjacent}} = 0$$

(in this case the opposite side shrinks to zero so that the hypotenuse overlaps with the adjacent side)

Therefore, **tan 0° = 0**

Example 9.5

Find the exact value of $2\sin 60° + \cos 30°$.

Solution

Using the values from the Table 9.1, it is noted that:

$$\sin 60^0 = \frac{\sqrt{3}}{2} \text{ and } \cos 30^0 = \frac{\sqrt{3}}{2}$$

$$2\sin 60^0 + \cos 30^0 = \left(2\times\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

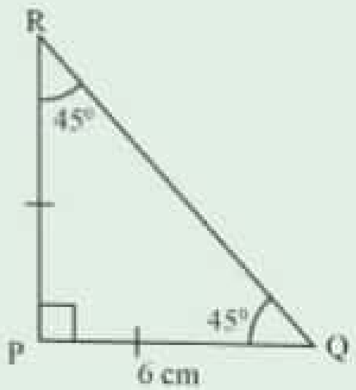
Therefore, $2\cos 60^0 + \cos 30^0 = \frac{3\sqrt{3}}{2}$.

Example 9.6

An isosceles triangle PQR is such that $P\hat{Q}R = 45°$ and $R\hat{P}Q = 90°$. If $\overline{PQ} = 6\text{cm}$, find the length of $\overline{RQ}$, using trigonometric ratios giving the answer in form of the radical or surd form.

Solution

Using the triangle PQR;



$$\cos 45° = \frac{6}{\overline{RQ}}$$

$$\overline{RQ} = \frac{6}{\cos 45°} = \frac{6}{\frac{\sqrt{2}}{2}}$$

$$= \frac{12}{\sqrt{2}} = 6\sqrt{2} \quad \text{(upon rationalizing the denominator).}$$

Therefore, $\overline{RQ} = 6\sqrt{2}$ cm.

Exercise 9.2

Answer the following questions:

1. Find the exact value of each of the following expressions:

   (a) $\sqrt{3}\tan 60^\circ$

   (b) $\sqrt{2}(\cos 45^\circ + \sin 45^\circ)$

   (c) $2(\sin 30^\circ + \cos 0^\circ)$

   (d) $\frac{\sqrt{3}}{3}\cos 30^\circ - 2\tan 45^\circ$.

2. If $\sin x = \frac{1}{2}$, find the following:

   (a) the value of x.

   (b) $\tan x$ given that x is an acute angle.

3. A ladder leans against a vertical wall and makes an angle 60° with the wall. If the highest point of the ladder is 4 m from the ground, find the length of the ladder.

4. Simplify each of the following:

   (a) $\cos 45^\circ \sin 45^\circ$

   (b) $\sin 60^\circ(\tan 30^\circ - \cos 30^\circ)$.

5. Find the value of each of the following:

   (a) $\frac{\sin 60^\circ}{\cos 60^\circ}$

   (b) $\frac{\sin 30^\circ}{\cos 30^\circ}$

6. Evaluate each of the following expressions:

   (a) $\frac{\sin 30^\circ + \cos 60^\circ}{2}$

   (b) $(\sin 60^\circ)^2 \cos 60^\circ$.

Trigonometric tables

Values of trigonometric ratios can be read from trigonometric table just as it is done with logarithms of numbers. Consider Table 9.2 which is a part of the larger table of sine values. The angle is read from the extreme left hand column. For example, to find sin 36°, locate 36° on the column of degrees and read the corresponding value under the column labeled 0′. This gives .5878. Thus, sin 36° = 0.5878. Similarly, to find sin 74° 50′, locate 74 on the extreme left column and in this row read the corresponding value under 48′.

Natural sine

Table 9.2: Trigonometric table

x	0′	6′	12′	18′	...	48′	54′	Mean Differences (Add)				
	0.0°	0.1°	0.2°	0.3°		0.8°	0.9°	1	2	3	4	5
0	0.0000	0.0017	0.0035	0.0052		0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227		0.0314	0.0332	3	6	9	12	15
⋮												
35	0.5736	0.5750	0.5764	0.5779		0.5850	0.5864	2	5	7	9	12
36	0.5878	0.5892	0.5906	0.5920		0.5990	0.6004	2	5	7	9	12
37	0.6018	0.6032	0.6046	0.6060		0.6129	0.6143	2	5	7	9	12
⋮												
73	0.9563	0.9568	0.9573	0.9578		0.9603	0.9608	1	2	2	3	4
74	0.9613	0.9617	0.9622	0.9627		0.9650	0.9655	1	2	2	3	4
75	0.9659	0.9664	0.9668	0.9673		0.9694	0.9699	1	1	2	3	4
⋮												

sin 74°48′ = 0.9650. The common difference is 50′ – 48′ = 2′ corresponding value under 2′ in the mean difference column is 2. So, add 2 to the last digit of 0.9650. Thus sin 74°50′ = 0.9652. Tangents of angles in the tangent table are read in exactly the same way as tables of sines. For cosine tables, the method is the same but the differences have to be subtracted.

For example, to find cos 64°13′, locate 64° in the extreme left column of the cosine tables (page 433) and read the corresponding value under 12′ which is 4352. Then read the corresponding value under 1 in the differences column, which is 3. Then subtract 3 from 4352 to obtain 4349. Thus, cos 64°13′ = 0.4349.

Example 9.7

By using trigonometric tables find the value of $\cos 16.4^\circ$.

Solution

Since $16.4^\circ = 16^\circ + 0.4^\circ$, locate 16° in the column x of the cosine table and read the corresponding value under column 4° which is 0.9593.

Therefore, $\cos 16.4^\circ = 0.9593$

Alternatively, convert 16.4° into degrees and minutes

$1^\circ = 60'$

$0.4^\circ = 24'$

Thus, $16.4^\circ = 16^\circ 24'$

Locate 16° in column x and read the corresponding value under the column of $24'$ which is 0.9593.

Converting 16.4° into degrees and minutes,

$16.4^\circ = 16^\circ 24'$

Therefore, $\cos 16.4^\circ = \cos 16^\circ 24' = 0.9593$

When the trigonometric ratio is known, the reverse process can be used to find the angle. For example, to find an angle whose cosine is 0.5348 look for the number in the middle section of the cosine table which is approximately equal to 5348. This corresponds to the angle $57^\circ\ 40'$.

Example 9.8

If $\tan x = 1.4073$ Find the value of x.

Solution

1.4071 corresponds to $54^\circ 36'$

Therefore, $x = 54^\circ 36'$.

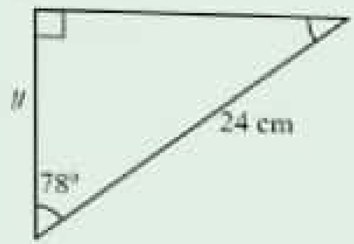
Note that, the values of sine and cosine cannot be greater than 1, but tangent values of angles can be greater than 1.

Example 9.9

Find the value of x and y in the following figures.

(a)

(b)



Solution (a)

$$\frac{x}{32\text{ cm}} = \sin 37^\circ$$

$$x = 32\text{ cm} \times \sin 37^\circ$$

From tables, $\sin 37^\circ = 0.6018$

Thus, $x = 32\text{ cm} \times 0.6018$

Therefore, $x = 19.2576\text{ cm}$.

Solution (b)

$$\frac{y}{24\text{ cm}} \quad \cos 78^\circ$$

$$y = 24\text{ cm} \times \cos 78^\circ$$

From tables, $\cos 78^\circ = 0.2079$

Thus, $y = 24\text{ cm} \times 0.2079$

Therefore, $y = 4.9896\text{ cm}$.

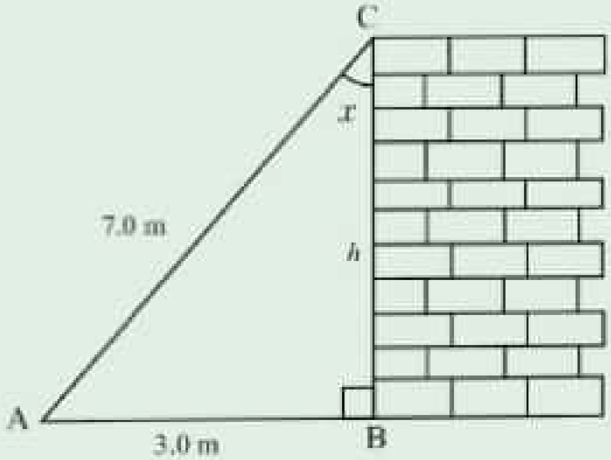
Example 9.10

A ladder 7.0 m long rests against a vertical wall so that the distance between the foot of the ladder and the wall is 3.0 m. Find the following:

(a) The angle which the ladder makes with the wall.

(b) The height above the ground at which the upper end of the ladder touches the wall.

Solution



Let $\overline{AC}$ represent the length of the ladder and $\overline{AB}$ the distance along the horizontal ground.

Then, from the figure, $\overline{AB} = 3$ m and $\overline{AC} = 7$ m.

(a) $\sin x = \frac{\overline{BA}}{\overline{AC}} = \frac{3.0}{7.0} = 0.4286$

$x = 25° \, 23'$ (reading from a sine table)

Therefore, the ladder makes an angle of 25° 23′ with the wall.

(b) $\cos 25°23' = \frac{\overline{CB}}{\overline{AC}}$

$= \frac{\overline{CB}}{7 \text{ m}}$

$\overline{CB} = \overline{AC} \cos 25°23'$

$= 7 \text{ m} \times 0.9034$ (reading from the cosine table)

$= 6.3238$ m

Therefore, the ladder reaches 6.32 m up the wall.

Note that, this could also be done by using Pythagoras' theorem (as the wall and horizontal ground meet at 90°).

Exercise 9.3

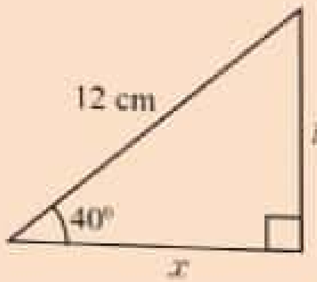
Anwer the following questions:

1. Use tables to find the values of each of the following:
   (a) $\sin 56^\circ$ (b) $\tan 36^\circ$ (c) $\cos 2^\circ$
   (d) $\cos 64^\circ 15'$ (e) $\sin 26^\circ 11'$ (f) $\tan 70^\circ$

2. Use tables to find the value of y.
   (a) $\cos y = 0.2034$ (b) $\sin y = 0.5975$ (c) $\tan y = 1.5000$
   (d) $\sin y = 0.8952$ (e) $\cos y = \frac{2}{5}$ (f) $\tan y = \frac{1}{3}$

3. By using trigonometric tables, find the values of x and y in each of the following triangles:

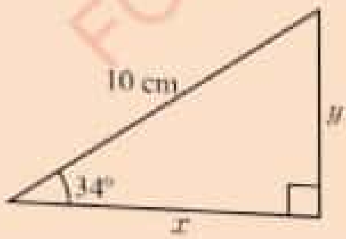
(a)



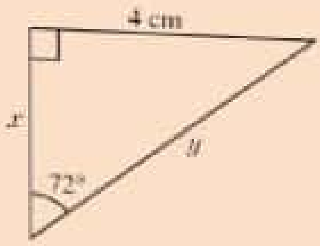
(b)

(c)

(d)



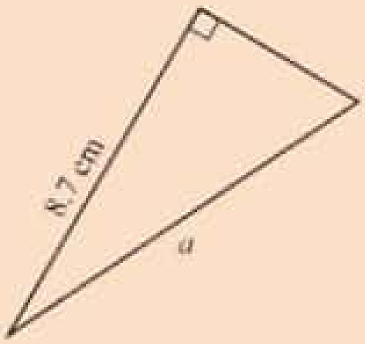
(e)



Math F2 – (BOOK) (DUMMY) Sept 4.indb 291

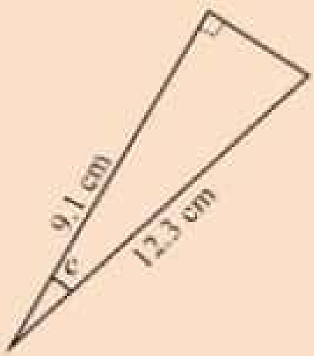
4. In each of the following triangles, find the length of the side or value of the angle marked with a letter in following figures.

(a)

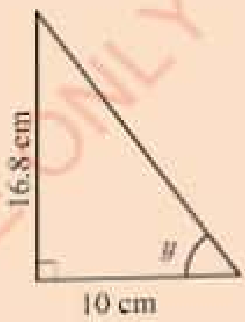


(b)

(c)

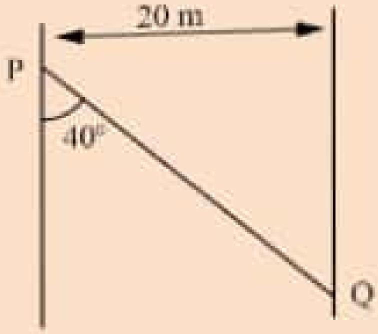


(d)

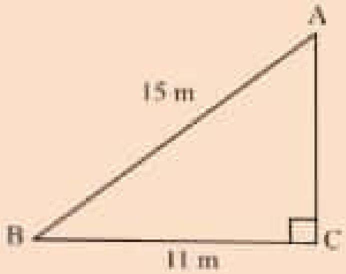


5. A water tap can be reached from Juma's house by walking 100 m North and then 30 m East. Find the bearing (direction) of the water tap from Juma's house.

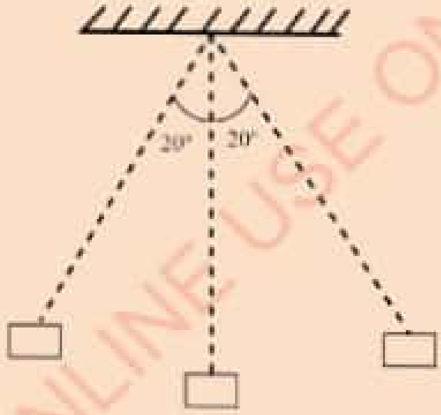
6. A river with parallel banks is 20 m wide. Find the distance $\overline{PQ}$ if P and Q are two points on either side of the river as shown in the following figure.



7. An inclined iron rod AB of 15 m long has a wire tied at A so as to be used to lift loads at point C which is 11 m from B as shown in the diagram. Find the angle that the iron rod makes with this wire.



8. A pendulum 30 cm long swings to and fro through an angle of 20° on either side. How high does the lower end of the pendulum rise?



9. The diagonal of rectangle makes an angle of 39° with the longer side. Find the width of the rectangle if its length is 50 cm.

Angle of elevation and angle of depression

When you are on top of a building, a hill or a tree, you need to lower your eyes from the horizontal position at a certain angle to see something down on the ground. In order to see an object above you, you need to raise your eyes through a certain angle.

The angles at which you need to raise or lower your eyes from a horizontal position are called **angles of elevation** and **angles of depression**, respectively.

Figure 9.7 illustrates the angles and positions relative to one another of two people standing at points A and B.

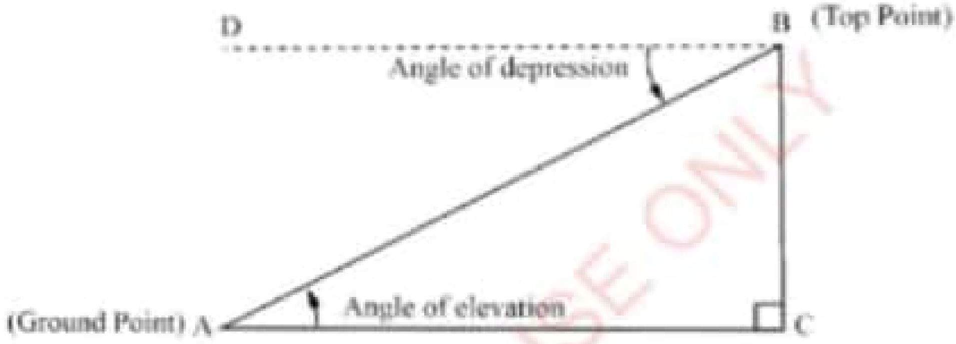


Figure 9.7: *Angles and positions relative to one another*

Angle $D\hat{B}A$ is called the angle of depression and angle $C\hat{A}B$ is called the angle of elevation. The line $\overline{AB}$ is the line of sight.

Consider an object situated at some point B (see Figure 9.8).

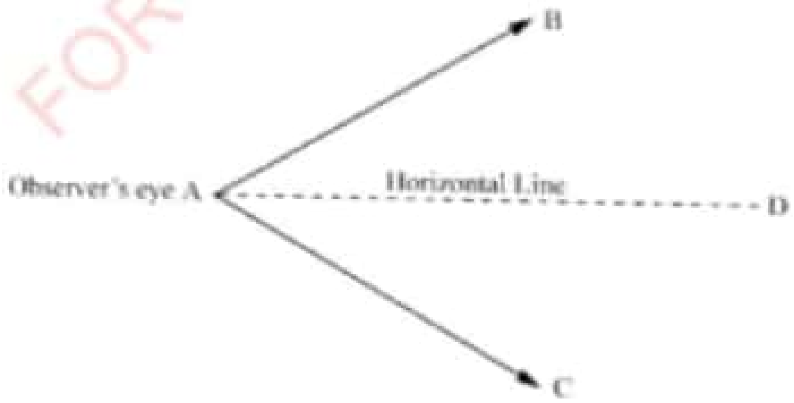


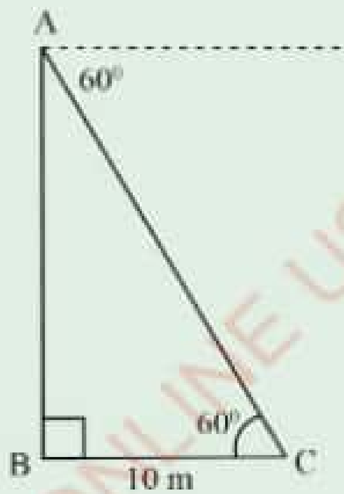
Figure 9.8: *Angles of elevation and depression*

The line joining the observer's eye at A is called the line of sight for B. The object to be observed may be above or below the horizontal line. If the object is above the horizontal line, then the angle between the line of sight and the horizontal is called the angle of elevation. If the object is below the horizontal, then it is the angle of depression. In Figure 9.8, $B\hat{A}D$ is the angle of elevation and $C\hat{A}D$ is the angle of depression.

Example 9.10

From the top of a tower, the angle of depression of a point on the ground 10m away from the base of the tower is 60°. How high is the tower?

Solution



Let A represent the top point of the tower, C is the point of observation and B is the base of the tower. Then $\tan 60° = \frac{\overline{AB}}{10 \text{ m}}$

$\overline{AB} = 10 \text{ m} \times \tan 60°$

$\overline{AB} = 10 \text{ m} \times 1.7321$

$\overline{AB} = 17.321 \text{ m}$

$\overline{AB} = 17.321 \text{ m}$

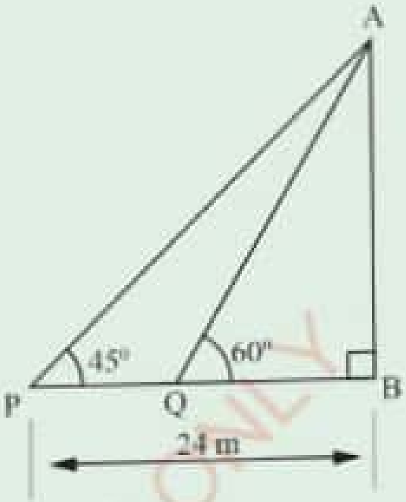
Therefore, the height of the tower is 17.32 m.

Example 9.11

Two pegs, P and Q, are on the level ground. Both pegs lie due West of a flag post. The angle of elevation of the top of the flag post from P is 45° and from Q is 60°. If P is 24 m from the foot of the flag post, find $\overline{PQ}$.

Solution

Let the height of the flag post be $\overline{AB}$.



$$\frac{\overline{AB}}{24\text{ m}} = \tan 45° = 1$$

$$\overline{AB} = 24\text{ m}$$

By using triangle AQB;

$$\frac{\overline{AB}}{\overline{QB}} = \tan 60° = \sqrt{3}$$

$$\frac{\overline{QB}}{\overline{AB}} = \frac{\sqrt{3}}{3}$$

$$\frac{\overline{QB}}{24\text{ m}} = \frac{\sqrt{3}}{3}$$

Hence, $\overline{QB} = 24\text{ m} \times \frac{\sqrt{3}}{3}$

$$= \frac{24\sqrt{3}}{3}\text{ m} = 8\sqrt{3}\text{ m}$$

It follows that:

$$\overline{PQ} = \left(24 - 8\sqrt{3}\right)\text{ m}$$

$$= 8\left(3 - \sqrt{3}\right)\text{ m}$$

$$= 10.144\text{ m}$$

Therefore, $\overline{PQ} = 10.14$ m.

Exercise 9.4

Answer the following questions:

1. A man whose eye is 180 cm above the ground is standing 8 m from a tree 7 m tall. What is the angle of elevation of the top of the tree from his eye?

2. The length of a shadow of a 16 m tree is 8 m. What is the size of the angle of elevation of the sun?

3. Find the height of a tower if the angle of elevation of the top is 34° at a point 20 m from the ground.

4. A tree casts a 60 m shadow when the angle of elevation of a sun is 25°. How tall is the tree?

5. The angle of elevation of the top of a tree from a point on the ground 30 m from the base of the tree is 37°. Find the height of the tree.

6. From the top of a cliff 8 m high, two boats are seen in a direction due West. Find the distance between the boats if their angles of depression from the top of the cliff are 45° an 30°. Find the actual distance of the boat which is further from the top of the cliff.

7. The angle of elevation of the top of a building 24 m high is observed from the top and from the bottom of a vertical ladder, and found to be 45° and 60°, respectively. Find the height of the ladder.

8. From a point P on the level ground the angle of elevation of the top of a flag post is 60°. If the height of the flag post is 39 m, how far from the base is the point P?

Chapter summary

1. Trigonometric ratios

$$\sin \hat{A} = \frac{\text{length of side opposite to angle } \hat{A}}{\text{hypotenuse}}$$

$$\cos \hat{A} = \frac{\text{length of adjacent side to angle } \hat{A}}{\text{hypotenuse}}$$

$$\tan \hat{A} = \frac{\text{length of side opposite to angle } \hat{A}}{\text{length of adjacent side to angle } \hat{A}}$$

2. Trigonometric ratios of special angles:

x	$\sin x$	$\cos x$	$\tan x$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

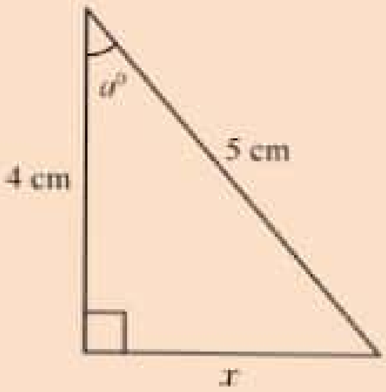
3. The values of trigonometric ratios can be read from trigonometric tables just as it is done with logarithms of numbers.

4. Angle of elevation and depression:
   (a) Angle of elevation: This angle is obtained when the line of sight is above the horizontal line.
   (b) Angle of depression: This angle is obtained when the line of sight is below the horizontal line.

Revision exercise 9

Answer the following questions:

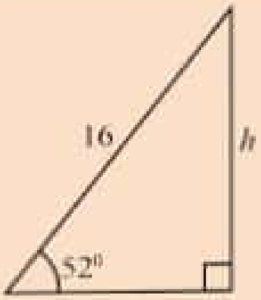
1. Use tables of trigonometric ratios to find each of the following:
   (a) sin 60° (b) cos 28° 28′ (c) tan 63° 48′
2. Use tables of trigonometric ratios to find the value of y in each of the following:
   (a) $\tan y = 0.9036$ (b) $\cos y = 0.2554$ (c) $\sin y = 0.4971$
3. In the following figure, find the values of a and x.



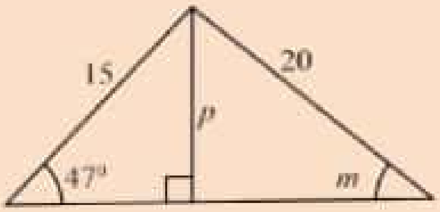
4. A pendulum consists of a bob hanging at the end of a string which is 18 cm long. Find the vertical height through which the bob rises and falls as the pendulum swings through 30° on each side of the vertical.
5. When the angle of elevation of the sun is 30°, the shadow of a post is 6 m longer than when it is 60°. Find the height of a post.
6. The angle of elevation of the top of a church tower from a point due East of it and 96 m away from its base is 30°. From another point due West of the church tower the angle of elevation of the top is 60°. Find the distanceof the later point from the base of the church tower.
7. Point A is 289 m from point C on a bearing N328°W, point B is 450 m from point C on a bearing N58°E. Find the distance from A to B.
8. When the angle of elevation of the sun in 55°, a tower casts a shadow 20 m long. Find the height of the tower.
9. A kite is flying directly over a straight path 100 m long. The angle of elevation of the kite from one end of the path is 35°, if the angle of elevation of the kite from the other end of the path is 55°, how high is the kite?
10. A boy finds that the angle of elevation of the top of a tree from a point on the ground is 25°. He walks in a straight line 30 m closer to the foot of the tree. The angle of elevation of the top is now 50°. How high is the tree?

11. Find the value of each unknown quantity shown in the following figures.

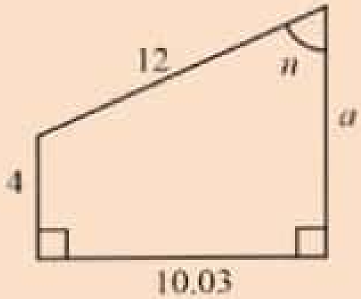
(a)



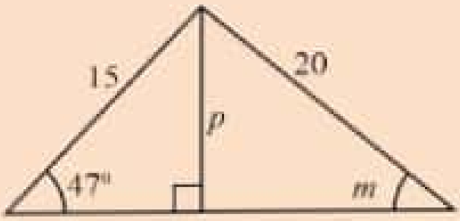
(b)



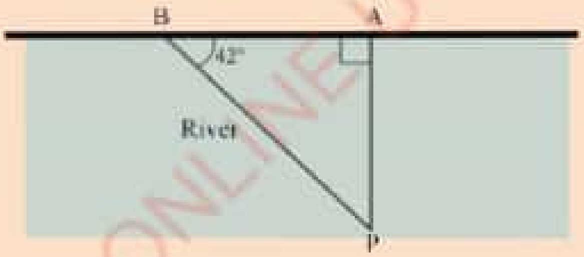
(c)



(d)



12. An observer is at point A on the bank of a river. The foot of a coconut tree is at point P directly across on the opposite bank. A distance AB of 27 m is measured along the bank so that $B\hat{A}P$ is a right angle and $A\hat{B}P = 42°$, as shown in the following figure.



Find the following:

(a) The width of the river.

(b) The height of a coconut tree if the angle of elevation of the top of the coconut tree is 22° from A.

(c) The angle of elevation of the top of the coconut tree from B.

(d) The distance from B to the top of the coconut tree.

13. A ladder of length 12 m is set against a vertical wall.

(a) If it makes an angle of 28° with the wall, how far up the wall does it reach?

(b) If it reaches 10 m up the wall, what angle does it make with the horizontal?

Chapter Ten

Sets

Introduction

The word set is used to describe a group of things with common characteristics such as tea cups, table mats, a pile of books, a collection of trees, a group of numbers and so on. A collection of things will qualify to be a set if the things have something in common like a group of students, set of tea cups and so on. In this chapter you will learn about description and types of sets, comparison of sets, subsets and universal sets. You will also learn some operations with sets, Venn diagrams and complement of sets. Competencies developed in this chapter will enable you to apply the concepts to simplify, categorise, analyse, and generalise different problems arising from real life situations in a more systematic way.

Activity 10.1: Collecting together items of the same characteristics

Materials required: Exercise books, pens, pencils

Steps:

1. Form a group of five members.
2. Each member of the group has to put an exercise book, a pen and a pencil on the table.
3. Arrange the items according to their specific characteristics.
4. How many groups of items do you have?
5. Name the groups by letters A, B and C.
6. What observations have you made after grouping the items?

Description of sets

A set is a collection of objects or things. Sets are denoted by capital letters. The objects contained in a set are called **elements** or **members** of the set. Members of a set are usually enclosed by curled brackets { }. For example, if B is a set of students in a class and John is a student in that class, then, John is a member of set B or John is an element of set B. The symbol $\in$ is used to denote 'a member of' or 'element of'.

This is denoted by John $\in$ B. Similarly, if $C = \{1, 2, 3\}$, then $1 \in C$, $2 \in C$ and $3 \in C$. The number of different elements in a set A is denoted by $n(A)$. For example, if $A = \{a, e, i, o, u\}$, then $n(A) = 5$.

Sets can be described in different ways. The most common ways used are by words, by listing (or roster form), and by a formula (or set builder notation).

For example, the set A of even numbers can be described as follows:

(a) By using words: A = {Even numbers}

(b) By listing: $A = \{2, 4, 6, 8, 10, ...\}$

(c) By formula: $A = \{x : x = 2n$, where $n \in N\}$ and is read as A is the set of all x such that x is an even number.

Example 10.1

Describe set A = {Odd numbers between 1 and 10} by:

(a) listing (b) formula

Solution

(a) $A = \{3, 5, 7, 9\}$

(b) $A = \{x : x = 2n - 1$, where $n \in \mathbb{Z}$ and $1 < n < 6\}$

Exercise 10.1

Answer the following questions:

In questions 1 to 3, list the elements of each set.

1. A = $\{x : x$ is an odd number $< 10\}$
2. B = {Days of the week which begin with the letter S}
3. C = {Prime numbers less than 13}

In questions 4 to 8, write the following sets in words.

4. A = {1, 2, 3, 4, ...}
5. B = {1, 3, 5, 7, 9, 11}
6. C = {a, e}
7. C = {2, 3, 5, 7, 11, 13, 17, 19}
8. E = {a, e, i, o, u}

In questions 9 to 12, write the given sets using the formula.

9. A = {Even numbers}
10. B = {Numbers divisible by 5}
11. C = {Natural numbers less than 20}
12. D = {Perfect squares}
13. State whether or not each of the following can be defined as a set, then give reasons.
    (a) {3, 5, 7, 11}
    (b) {Dog, cow, stone, goat}
    (c) {a, e, i, o, u}
    (d) {Asia, Africa, Europe, Australia}
    (e) {January, Monday, Wednesday, November}
14. List the elements of each of the following sets:
    (a) P = {Last three consonants}
    (b) D = {Months begins with the letter J}
    (c) F = {Prime numbers less than 20}

Types of sets

Sets can be categorised as either finite or infinite.

Finite set

A set is finite if it has a known number of elements. For example, A = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24}is a finite set because it has 12 elements.

Infinite set

A set is infinite if it has unknown number of elements. That is, the counting process of its elements does not end. For example, B = {2, 4, 6, 8, ...} is an infinite set because all elements of this set cannot be counted. When defining an infinite set, list a few elements of the set followed by three dots (…) to show that the list of elements in the set continue.

Empty set

An empty set is a set which has no elements. For example, the set of prime numbers between 31 and 37 is an empty set because there is no prime number between 31 and 37. Also, the set of lions studying in your school is an empty set because lions do not go to school. An empty set is denoted by the symbol { } or Ø.

Comparison of sets

When two or more sets are compared, they may be equivalent, equal or one is a subset of the other. For example, the sets A = {2, 4, 6, 8} and B = {a, b, c, d} are said to be equivalent because they have equal number of elements.
That is, n(A) = n(B) = 4.
Generally, if n(A) = n(B), then set A and B are equivalent.

Example 10.2

Show that A = {4, 6} and B = {1, 7} are equivalent.

Solution

n (A) = 2

n (B) = 2

Therefore, set A and set B are equivalent.

Sets A and B are equivalent because they have equal number of elements. That is, n(A) = n(B) = 2.

Two sets A and B are said to be equal if they are equivalent and their members are alike regardless of their arrangement or order. For example, if A = {1, 2, 3, 4} and B = {3, 1, 4, 2} then A = B since all elements of A are elements of B and all elements of B are also elements of A.

Note that, all equal sets are equivalent but not all equivalent sets are equal.

Exercise 10.2

Answer the following questions:

1. Which of the following sets are finite, infinite or empty?
   (a) A = {Nairobi, Dar es Salaam}
   (b) B = {2, 4, 6, ..., 36}
   (c) C = $\{x : x \in N;\ x > 0\}$
   (d) D = {All lions in your class}
   (e) E = {All mango trees in the world}
   (f) F = {x : x is a student aged 100 years in your school}
   (g) G = {x : x is a prime numbers}
   (h) H = {1, 3, 5, 7}
   (i) I = { }

2. Which of the following pairs of sets are equivalent and which are not eqivalent?
   (a) A = {a, b, c} and B = {b, c, d}
   (b) B = {Rufiji, Ruaha, Malagarasi} and C = {Lion, Leopard}
   (c) D = {a, b, c, d} and E = {a, b, c}

3. Which of the following sets are equal?
   (a) A ={a, b, c, d}; B = {d, a, b, c}; C = {a, e, i, o, u}
   (b) D = {a, b, c, d, e}; E = {d, c, b, a}; F = {a, e, b, c, d}

4. State whether the following are finite, infinite or empty sets:
   (a) A = {African countries in Asia continent}
   (b) B = {All the letters of the English aphabets}
   (c) D = {All months with 32 days}
   (d) E = {2, 4, 6, 8, 10}
   (e) F = $\{x : x > 20\}$
   (f) G = {1, 2, 3, 4, 5, ...}

5. Write three examples each of finite, infinite, and empty set in a real life situation.

Subsets

Given two sets A and B, B is said to be a subset of A if all elements of B belong to A. For example, if A = {a, b, c, d, e} and B = {a, b, c, e} then B is a subset of set A because all elements of B belong to A. If B has less elements than A, then B is called a **proper subset** of A, and is denoted by $B \subset A$. However, if B is a subset of A and the two sets are equal then B is called an improper subset of A, and is written as $B \subseteq A$. Note that, the empty set is a subset of any set.

Example 10.3

List all the subsets of A = {a, b}

Solution

The subsets of A = {a, b} are all possible sets which can be obtained by taking some elements of set A or all elements of A.

Therefore, the subsets of set A are: { }, {a}, {b}, {a, b}.

Example 10.4

What are the subsets of B = {1, 2, 3}?

Solution

The subsets are: { }, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}

The number of subsets of a given set can be calculated using a formula. The formula is obtained by observing the pattern of forming subsets of sets with increasing number of elements as shown in the following table.

Set	Subsets	Number of subsets
{ }	{ }	$1 = 2^0$
{1}	{ }, {1}	$2 = 2^1 = 2$
{2, 3}	{ }, {2}, {3}, {2,3}	$4 = 2 \times 2 = 2^2$
{1, 2, 3}	{ }, {1}, {2}, {3}, {1,2}, {1,3}, {2, 3}, {1, 2, 3}	$8 = 2 \times 2 \times 2 = 2^3$
{1, 2, 3,…,*n*}	{ }, {1}, {1, 2},…, {1, 2,…,*n*}	2^n

Therefore, the formula for finding the number of subsets of a set with n elements is 2^n.

Example 10.5

How many subsets are in set A = {1, 2, 3, 4}

Solution

Since n(A) = 4, then $2^n = 2^4 = 16$

Therefore, set A has 16 subsets.

Universal set

A set which contains the elements of all sets under consideration is called the universal set. It is denoted by U. For example, the set of integers contains all elements of sets such as odd numbers, prime numbers, even numbers, counting numbers and whole numbers. In this context, the set of integers is considered as the universal set.

Exercise 10.3

Answer the following questions:

1. List all the subsets of each of the following sets:
   (a) A = {1} (b) B = Ø (c) C = {Tito, Juma}

2. How many subsets has each of the following sets?
   (a) A = {2, 4, 6, 8} (b) B = {a, b, c, d, e, f, g} (c) { }

In questions 3 to 5, which set in each pair is a subset of the other? Use the symbol ⊂.

3. A = {a, b, c, d, e, f, g, h} and B = {d, e, f}
4. A = {2, 4} and D = {2, 4, 5}
5. A = {1, 2, 3, 4, ...} and C = {2, 4, 6, 8, ...}
6. How many subsets are there in a set with seven elements?

7. For each of the following statements write **T** if the statement is true and **F** if the statement is false:

   (a) If G = {4, 5, 6} and H = {5, 6, 7, 8}, then $G \subset H$
   (b) If K = { } and L = {tree, house, egg}, then $L \subset K$
   (c) If A = {s, t, u, v} and B = {s, t, u, v}, then $A \subseteq B$
   (d) If I = {1, 2, 3, 4, ...} and J = {4, 6, 8, ...} $J \subset I$

8. If G = {cities, towns and regions of Tanzania} which of the following sets are subsets of G?

   A= {Nairobi, Dar es Salaam}
   B = {Dodoma, Mombasa, Mwanza}
   C = Ø
   D = {Arusha, Iringa, Bagamoyo}
   E = {Mbeya, Tunduru, Ruvuma}

9. Which of the following sets are subsets of K, where, K = {p, q, r, s, t, u, v, w}

   (a) A = {p, s, t, x}
   (b) B = {q, r, d, t}
   (c) C = { }
   (d) D = {p, q, r, s, t, u, v, w}
   (e) E = {a, b, c, d}
   (f) F = {s, v, q}

10. What is n(A) if:

    (a) A = { } (b) A = {2, 3, 4, 5, 6}

11. Write in words the universal set of the following sets:

    (a) A = {a, b, c, d} (b) B = {1, 2, 3, 4}

12. How many subsets are there in each of the following sets?

(a) A = {1, 2, 3, 4, 5} (b) B = {a, b, c} (c) C = {a, e, i, o u}

13. Using set notation, show which set is subset of the other:

(a) A = {Countries in East Africa} and B = {Tanzania, Kenya, Uganda}

(b) C = {p, q, r, s, t, u, v} and D = {s, t, u, v}

Operations with sets

Two or more sets can be combined together to form one set under given conditions.

Activity 10.2: Identifying union of sets

In pairs or groups perform the following:

1. Each member has to list any five counting numbers.
2. Collect all the numbers together to get a large collection of numbers.
3. Count the numbers in the large collection.
4. What is your observation?

Union

The union of two sets A and B is the set that contains all the elements from each set without repetition of an element. It is denoted by symbol "$\cup$", hence A union B is written as $A \cup B$ which means, $x \in (A \cup B)$ if $x \in A$ or $x \in B$ (or both).

For example, if A = {2, 4, 6} and B = {2, 3, 5}

then, $A \cup B = \{2, 4, 6\} \cup \{2, 3, 5\} = \{2, 3, 4, 5, 6\}$

Example 10.6

Find A∪B if A = {a, b, c, d, e, f} and B = {a, e, i, o, u}.

Solution

A∪B if A = {a, b, c, d, e, f, i, o, u}.

Intersection

Activity 10.3: Identifying the intersection of sets

Materials required: A piece of paper and a pen.

1. Each student has to write down all natural numbers from 1 to 10, then name it as group A.
2. Write down all even numbers from group A, then name them group B.
3. Write down all prime numbers from group A, then name them group C.
4. Write down the numbers in group B and group C.
5. Is there any number which is in both group B and group C?

The intersection of sets A and B is the set which contains all the elements found in both sets. It is denoted by a symbol "∩". Thus, set A intersection B is written as A∩B, which means, $x \in (A \cap B)$ if $x \in A$ and $x \in B$.

For example, if A = {1, 2, 3, 4, 5} and B = {1, 3, 5} then A∩B = {1, 3, 5}.

Sets which have common elements are called joint sets. Sets which have no common elements are called disjoint sets.

Example 10.7

Find A∩B if A = {a, e, i, o, u} and B = {a, b, c, d, e, f} and state if A and B are joint or disjoint sets.

Solution

A∩B = {a, e}

Therefore, A and B are joint sets.

Example 10.8

Find $A \cap B$ if $A = \{a, e, i\}$ and $B = \{b, c, f\}$ and state if A and B are joint or disjoint sets.

Solution

$A \cap B = \emptyset$

Therefore, since they have no element in common, then A and B are disjoint sets.

Complement of a set

If A is a subset of a universal set, then the members of the universal set which are not in A form the complement of A. The complement of set A is denoted by A' or A^c. For example, if $U = \{a, b, c, d, ..., z\}$ and $A = \{a, b\}$, then $A' = \{c, d, e, ..., z\}$.

Example 10.9

Given $U = \{15, 45, 135, 275\}$, $A = \{15\}$. Find A'.

Solution

$A' = \{45, 135, 275\}$

Example 10.10

Given $U = \{a, e, i, o, u\}$ and $B = \{e, i\}$. Find B'.

Solution

$B' = \{a, o, u\}$

Example 10.11

Given; U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}
A = {2, 3, 5, 7, 11, 13} and B = {2, 4, 6, 8, 10, 12, 14}.

Find: (a) A′∪B′ (b) A∩B′

Solution

A′ = {1, 4, 6, 8, 9, 10, 12, 14}

B′ = {1, 3, 5, 7, 9, 11, 13}

(a) A′∪B′ = {1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}

(b) A∩B′ = {3, 5, 7, 11, 13}

Exercise 10.4

From question 1 to 10, find the union and intersection of the given pairs of sets.

1. A = {5, 10, 15}, B = {15, 20}
2. A = { }, B = {14, 16}
3. A = {first five letters of the English alphabet}, B = {a, b, c, d, e}
4. A = {counting numbers}, B = {prime numbers}
5. A = {cup, spoon}, B = {cup}
6. A = {All multiples of 5 less than 30}, B = {All multiples of 10 less than 30}
7. A= {All prime factors of 42}, B = {All prime factors of 15}
8. A = {All even numbers less than 10}, B = {All multiples of 3 less than 12}
9. A = {64, 81, 100, 121} and B = {64, 81, 144}
10. A = {a, b, c, d}, B = {d, e} and C = { }
11. What is the name given to sets which have no common elements?
12. Find the intersection of the following sets: A = {a, b, c, d}, B = {1, 2, 3, 4}

13. If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 7\}$, find A' and B'.

14. If U = {mango, orange, tomato, cabbage}, A' = {mango, tomato}, list the elements of A.

15. If $A = \{1, 3, 5, 7, 8\}$ and $B = \{2, 4, 6, 8, 10\}$ list the element of $A \cup B$.

16. If U = {red, green, blue, black, white}, find A' if A = {green, white, black}.

17. Given that: $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$. Find the following:
    (a) $A' \cap B$ (b) $A \cap B'$
    (c) $B' \cup A'$ (d) $B \cup A'$

18. Given the universal set $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$. If $A = \{0, 1, 3\}$ and $B = \{5, 6, 7\}$ write T for a true statement and F for a false statement in each of the following:
    (a) $A' = B$ (b) $A' \cup B' = \{1, 2, 3, 4, 5, 6, 7\}$
    (c) $A' \cap B' = (A \cup B)'$ (d) $n(A \cap B) = 1$
    (e) $n(A' \cup B') = 9$

19. For each of the following pairs of sets, find their union and intersection:
    (a) $A = \{2, 4, 6\}$, $F = \{4, 6, 8, 10\}$
    (b) $Y = \{x : 2 < x \leq 8\}$, $W = \{x : 7 \leq x < 11\}$

20. Given that $U = \{a, b, c, d, e, f\}$, $A = \{a, b, c\}$ and $B = \{a, d, e, f\}$. Find each of the following:
    (a) $A \cap B$ (b) $A \cup B$ (c) $A' \cap B$
    (d) $A \cap B'$ (e) $A' \cap B'$ (f) $A' \cup B'$

Venn diagrams

A set can be clearly represented by a drawing called a **Venn diagram** which was introduced by an English Mathematician named John Venn. He used ovals to represent sets. In constructing a Venn diagram, ovals are inscribed in a rectangle which stands for a universal set. For example, A = {a, b, c} could be represented in a Venn diagram as shown in Figure 10.1.

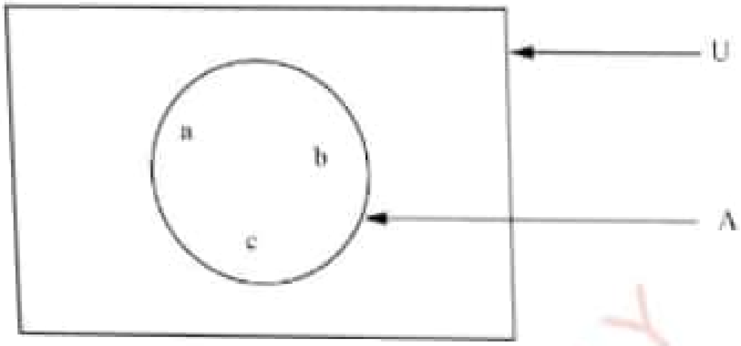


Figure 10.1: *Venn diagram for set A*

U is the universal set which in this case, is the set of letters of the English alphabet. A is a subset of U. If sets have elements in common, the ovals overlap. Sets which have common elements are called **joint sets**. For example, if A = {a, b, c} and B = {a, b, d} then the relation between A and B is represented as follows:

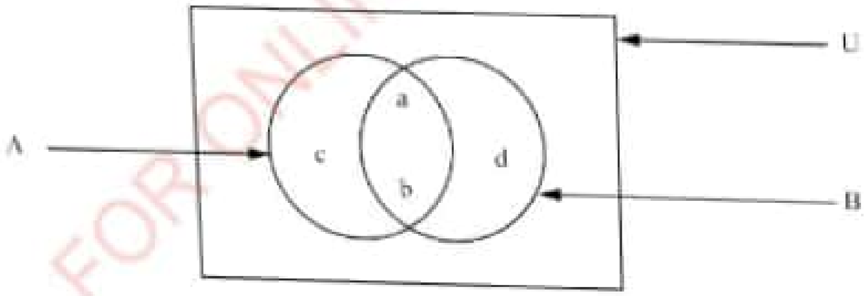


Figure 10.2: *Joint sets in a Venn diagram*

If the sets do not overlap in a Venn diagram, they are called **disjoint** sets. For example, if A = {a, b} and B = {1, 2} the relation between A and B is shown in Figure 10.3.

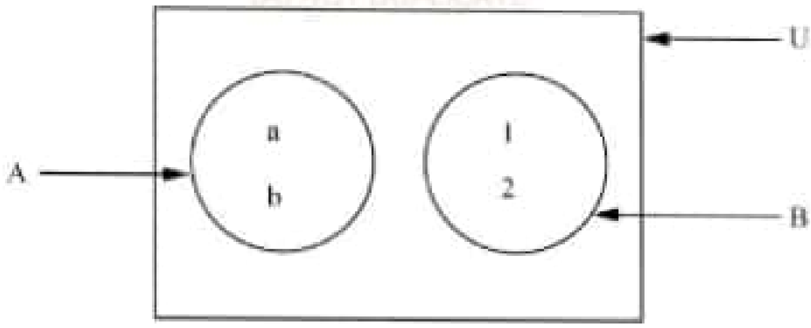
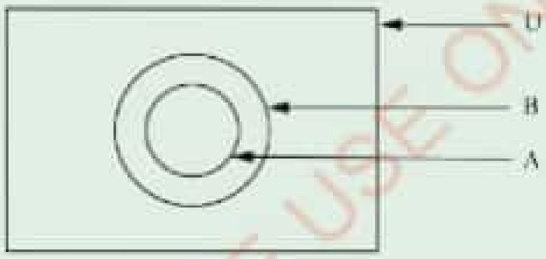


Figure 10.3: *Disjoint sets in a Venn diagram*

Example 10.12

If A is a proper subset of B, represent the two sets in a Venn diagram.

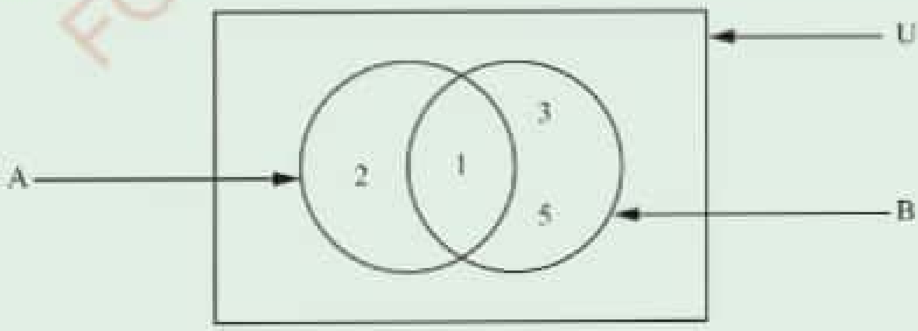
Solution



Example 10.13

Represent A∪B in a Venn diagram if A = {1, 2} and {1, 3, 5}.

Solution



Example 10.14

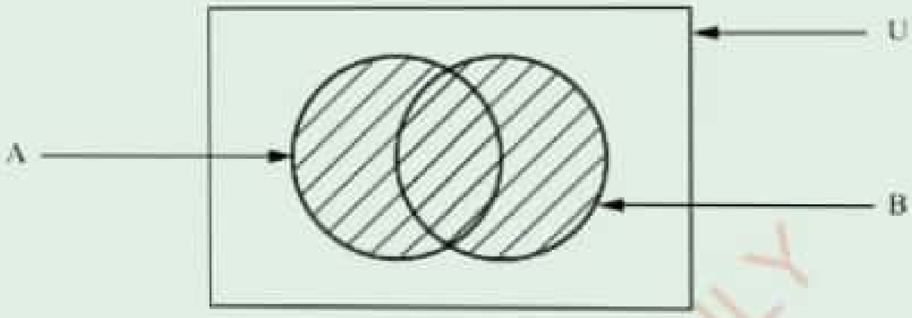
If set A and B have some elements in common, represent:

(a) $A \cup B$

(b) $A \cap B$ in a Venn diagram and shade the required area

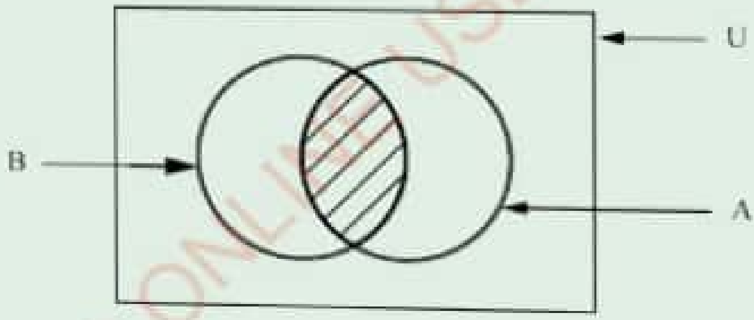
Solution

(a) $A \cup B$



Shaded region is $A \cup B$

(b) $A \cap B$



Shaded region is $A \cap B$

Number of elements in the combined set

Calculation of the number of elements in combined sets can be done with the aid of Venn diagrams by inserting digits for the number of elements in appropriate regions.

Example 10.15

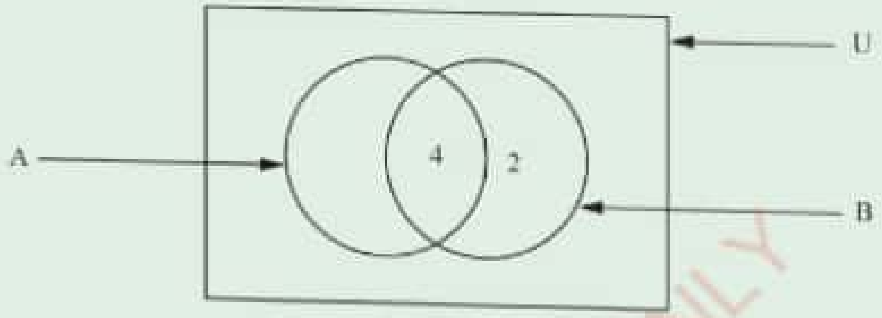
If A and B are two sets such that $n(A \cap B) = 4$, $n(A \cup B) = 6$ and $n(A) = 4$:

(a) How many elements are in B?

(b) Which set is subset of the other?

Solution

Using a Venn diagram, the two sets can be represented as follows:

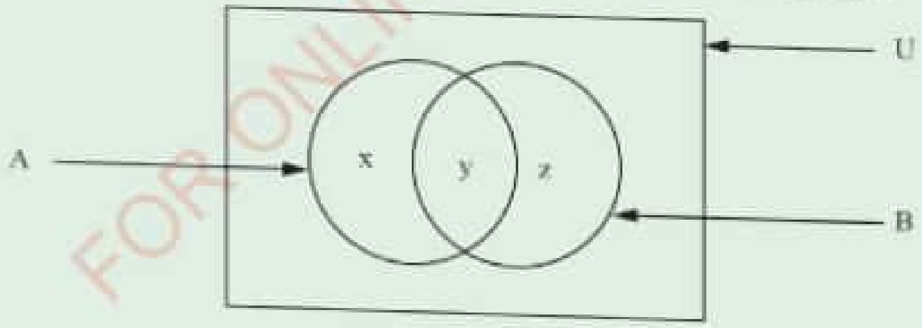


(a) Elements of B only = 6 − 4 = 2

Therefore, B has 4 + 2 = 6 elements.

(b) $A \subset B$. Because all elements of set A are contained in set B.

In general, the number of elements in two sets is connected by the formula: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, which is established as follows:



Let the number of elements in each region be x, y and z as shown in the figure. Then, the following expressions can be formulated:

$n(A) = x + y$

$n(B) = z + y$

$n(A \cap B) = y$

$n(A \cup B) = x + y + z$

$n(A) + n(B) = (x+y) + (y+z)$

$n(A) + n(B) = (x+y+z) + y$

$n(A) + n(B) = n(A \cup B) + n(A \cap B)$

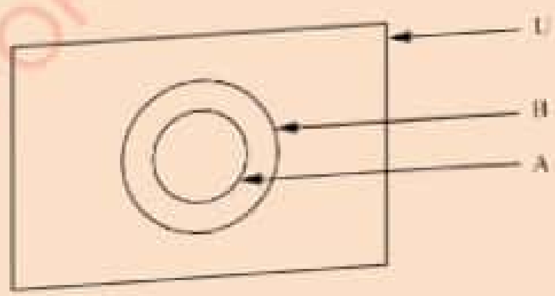
Therefore, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Exercise 10.5

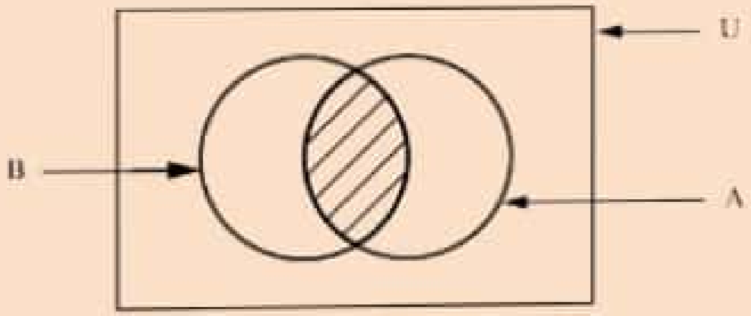
Answer the following questions:

1. Represent the following sets in Venn diagrams:
   (a) $A = \{a, b, c, d\}$
   (b) $A \subset B$
   (c) $A = \{a, b, c\}$ and $B = \{a, b, c\}$
   (d) $A = \{1, 2, 3\}$ and $B = \{4, 6, 8\}$

2. Write in words the relationship between the two sets shown in the following figure.



3. Describe in set notation the meaning of the shaded regions in the following Venn diagram.



4. Draw Venn diagrams and shade the regions represented by each of the following:

   (a) $A \cup B$ (b) $A \cap B$ (c) $(A \cap B) \cap C$

5. If $n(B) = 4$, $n(A \cup B) = 4$ and $n(A \cap B) = 0$, how many members are in set A?

6. If $n(A) = 8$, $n(A \cap B) = 8$, what is the relationship between A and B?

7. If $n(A \cap B) = 6$, $n(A \cup B) = 10$ and $n(B) = 6$

   (a) How many members are in set A?

   (b) Which set is a subset of the other?

8. Suppose $n(A \cap B) = 0$ and $n(A \cup B) = 4$. What is the relationship between A and B?

Representation of the complement of a set by a Venn diagram

If A is a subset of the universal set U, then the complement of set A may be represented by a Venn diagram as shown in Figure 10.4 where A′ is a shaded region.

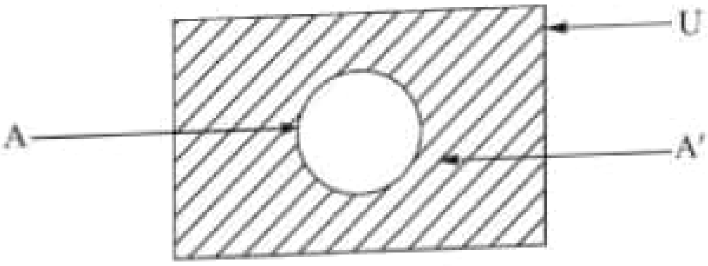


Figure 10.4: *Venn diagram showing complement of a set.*

Example 10.16

Given that U = {*a, b, c, d, e*} and A = {*b, c, e*}

(a) Represent the information in a Venn diagram

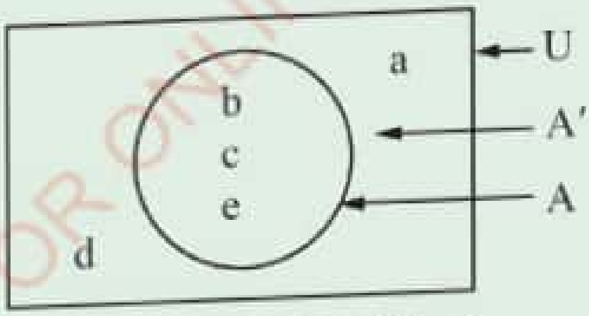
(b) Determine n(A), n(A′) and n(U)

(c) Find n(A) + n(A′)

(d) Comment on the result of n(U) and n(A) + n(A′)

Solution

(a)



(b) n(A) = 3

n(A′) = 2

n(U) = 5

(c) n(A) + n(A′) = 5

(d) The number of elements in a universal set is equal to the sum of the number of set A and its complement

Example 10.17

Given the following sets $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{3, 4, 5\}$ and $B = \{1, 2, 4, 6\}$. Show that $(A \cup B)' = A' \cap B'$ by using Venn diagram.

Solution

Required: To show that $(A \cup B)' = A' \cap B'$ using Venn diagram.

The given sets are $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{3, 4, 5\}$ and $B = \{1, 2, 4, 6\}$.

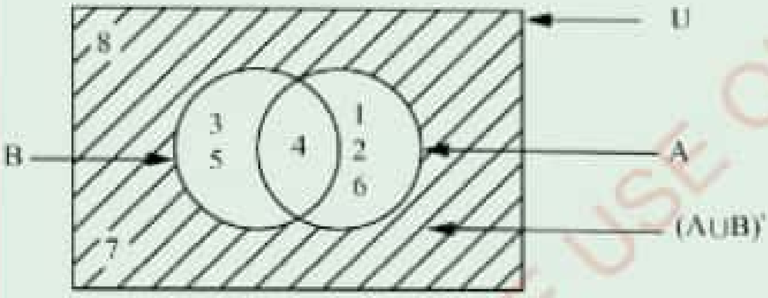
The complement of the two sets A and B are;

$A' = \{1, 2, 6, 7, 8\}$ and $B' = \{3, 5, 7, 8\}$

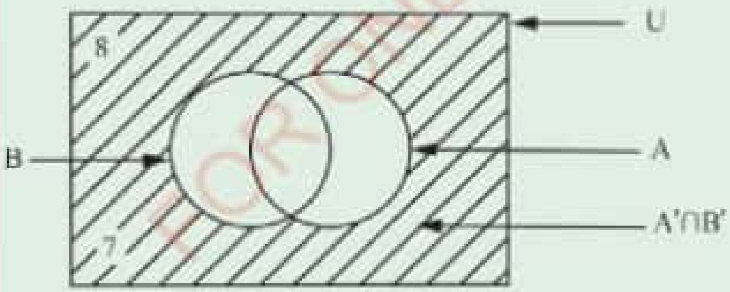
LHS: $(A \cup B)'$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$(A \cup B)' = \{7, 8\}$



RHS: $A' \cap B' = \{7, 8\}$



Therefore, $(A \cup B)' = A' \cap B'$.

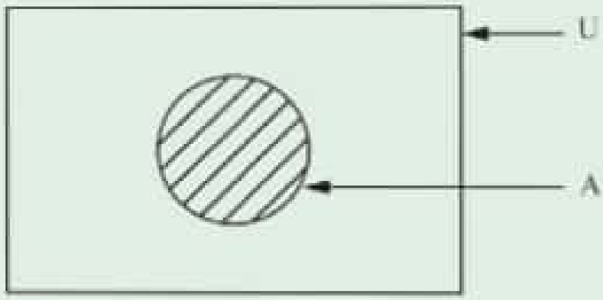
Since, double shaded area satisfies the equation.

Therefore, $(A \cup B)' = A' \cap B'$

Example 10.18

Represent $A \cap U$ in a Venn diagram and shade the required region.

Solution



Exercise 10.6

Answer the following questions:

1. Assuming that A' and B' have some elements in common, draw a Venn diagram to show $A' \cap B$ and shade the required region.
2. If A and B are joint sets, represent $A' \cup B'$ in a Venn diagram. Shade the required region.
3. In a Venn diagram, shade the region representing $A \cup A' = U$.
4. If A = {All even numbers} and B= {All perfect squares}; draw a Venn diagram to illustrate the relationship between A and B.
5. If $n(A \cup B) = 30$, $n(A) = 14$ and $n(A \cap B) = 6$, use a Venn diagram to find $n(B)$.

Word problems

Example 10.19

In a class of 120 students, 40 learn English, 60 learn Kiswahili and 30 learn both Kiswahili and English. How many students learn:

(a) English only?

(b) Neither English nor Kiswahili?

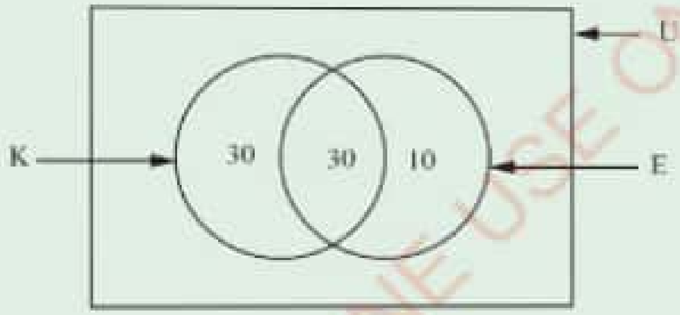
Solution

Let U = {All students in the school}

E = {All students who learn English}

K = {All students who learn Kiswahili

Representing the three sets in a Venn diagram gives:



Since $n(E \cap K) = 30$

$n(E) = 40$

$n(K) = 60$

Number of students learning English only $= n(E) - n(E \cap K)$

$= 40 - 30 = 10$

Number of students learning Kiswahili only $= n(K) - n(E \cap K)$

$= 60 - 30 = 30$

(a) 10 students learn English only.

(b) From the diagram those who learn English or Kiswahili or both are 70. Therefore, those who learn neither Kiswahili nor English are:

$120 - 70 = 50$ students

Alternative method

The number of students who learn either English or Kiswahili is $n(E \cup K)$.

The number of those who learn both English and Kiswahili is $n(E \cap K)$

Thus,

$$n(E \cup K) = n(E) + n(K) - n(E \cap K)$$

$$= 40 + 60 - 30 = 70$$

(a) Those who learn English only = $n(E) - n(E \cap K) = 40 - 30 = 10$ students.

(b) $n(E \cap K)' = n(U) - n(E \cup K) = 120 - 70 = 50$

Therefore, the number of students who learn neither English nor Kiswahili is 50.

Example 10.20

In a certain school, 50 students eat meat, 60 eat fish and 25 eat both meat and fish. Assuming that every student eats meat or fish, find the total number of students in the school.

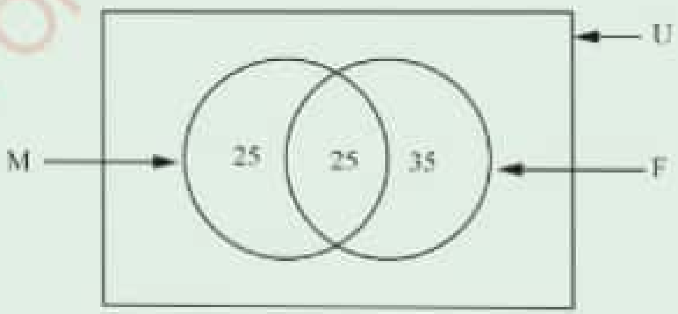
Solution

Let U = {All students in the school}

M = {All student who eat meat}

F = All students who eat fish}

By use of Venn diagrams, this gives



There are 85 students.

Alternative method

$$n(M \cup F) = n(M) + n(F) - n(M \cap F)$$
$$= 50 + 60 - 25$$
$$= 85 \text{ students}$$

Therefore, the total number of students in the school is 85.

Exercise 10.7

Answer the following questions:

1. In a class, 15 students play basketball, 11 play netball and 6 play both basketball and netball. How many students are there in the class if every student plays at least one game?
2. In a school of 160 students, 50 have bread for breakfast and 80 have sweet potatoes. How many students have neither bread nor potatoes for breakfast assuming that none takes both bread and sweet potatoes?
3. In a class of 20 students, 12 students study English but not History, 4 study History but not English, and 1 studies neither English nor History. How many students in the class study History?
4. There are 24 men at a meeting, 12 are farmers, 18 are soldiers, and 8 are both farmers and soldiers.
   (a) How many men are either farmers or soldiers?
   (b) How many men are neither farmers nor soldiers?
5. In a class of 30 students, 20 students take Physics, and 12 take both Chemistry and Physics. How many students take Chemistry, if 8 students take neither Physics nor Chemistry?
6. At a certain meeting 30 people drank Pepsicola, 60 drank Coca – Cola and 25 drank both Pepsicola and Coca-Cola. How many people were at the meeting assuming that each person took Pepsicola or Coca-Cola?
7. Every woman in a certain club owns a landrover or a bicycle. If 23 women own landrovers, 14 own bicycles and 5 own both a landrover and a bicycle, how many women are there in the club?
8. A survey of 160 households, each of whom kept a cat or a dog or both, showed that 95 kept cats and 80 kept dogs. How many kept both a cat and a dog?

9. Among 90 men who had spent a day in volleyball or football it was observed that 60 played volleyball and 22 played both volleyball and football. How many men played football?
10. In a village, all the people speak Kiswahili or English or both. Among them, 97% speak Kiswahili and 64% speak English.
    (a) What percentage speaks both Kiswahili and English?
    (b) What percentage speaks English only?
11. In a class of 36 students, everyone studies Biology or Physics or both. If 9 students study both subjects and 12 study Physics but not Biology, how many students study Biology but not Physics?

Chapter summary

1. A set is a group or collection of things or objects that have common characteristics.
2. A set is finite if it has a known number of elements.
3. A set is infinite if it has an unknown number of elements.
4. An empty set is a set which has no elements.
5. Equivalent sets have equal number of elements.
6. Equal sets have the same elements and an equal number of elements.
7. A subset contains some of the elements or members of another set.
8. A complement set consists of all members which are not in the given set, but are elements in a universal set.
9. Universal set contains all elements of all sets under consideration.
10. A Venn diagram is a drawing used to give a pictorial representation of sets or relationships between sets.
11. Union of two sets is a set which is formed when the members of the two sets are collected together without repetition.
12. Intersection of two sets is a set formed by taking all elements that are common in both sets.

Revision exercise 10

Answer the following questions:

1. If A = {a, b, c}, which of the following statements are true?
   (a) A∈A (b) d∈A (c) c∈A (d) b∈A

2. If A = {a, b, c, d} which of the following statements represent set A?
   (a) A set of four letters of the English alphabet.
   (b) Some of the sets of the first four letters of the English alphabet.
   (c) The set of the first four letters of the English alphabet.
   (d) The set of the four consonants of the English alphabet.

3. Describe the following set in words A = {1, 4, 9, 16, 25}

4. List the elements of the set given by A = {x : x is a counting number < 8}

5. Which of the following sets is an empty set?
   (a) A = {Ø} (b) B = {0} (c) R = Ø (d) S = { }

6. Which of the following sets are equal or equivalent?
   A = {1, 2, 3, 4, 5}
   B = {1, 4, 7, 10, 15}
   C = {cars, ships, aeroplanes}
   D = {cars, ships, aeroplanes}

7. Which of the following sets are finite or infinite?
   A = {x : x is a prime number}
   B = {all letters of the English alphabet}
   C = {all students in your school}
   D = {1, 4, 9, 16,...}
   E = {1, 4, 9, 16, 25,..., 81}

Math F2-(BOOK) (DUMMY) Sept 4.indb 327

8. If A = {a, b} show by using symbols that a and b are elements of A.

9. How many subsets are there in A = {a, b, c, d, e, f, g}

10. List all the subsets of A = {2, 4, 6}

11. In the following pairs of sets name the set which is a subset of the other:

(a) A = {a, b, c} and B = {e, d, b, c, a}

(b) C = {Tom, John, Idrisa} and D = {John, Tom}

12. If A = {a, b, c}, B = {4, a, 1} and D = {b, a} which of the following statements are true?

(a) $A \subset B$ (b) $B \subset D$ (c) $A \subset D$ (d) $D \subset A$

13. If A = {a, b, c, d} and B = { } find:

(a) $A \cup B$ (b) $A \cap B$

14. Let A be the set of the first ten counting numbers and B be the set of the first four prime numbers. Find:

(a) $A \cup B$ (b) $A \cap B$

15. If U = {1, 2, 3, 4,5} and A′ = {2, 3, 5} list the elements of A.

16. If U = {a, b, c, d, e}, B = {c, d} and A = {a, b, c} find the following:

(a) $A' \cap B'$ (b) $A \cup (B \cap A)$

17. Draw a Venn diagram and show by shading the region representing each of the following:

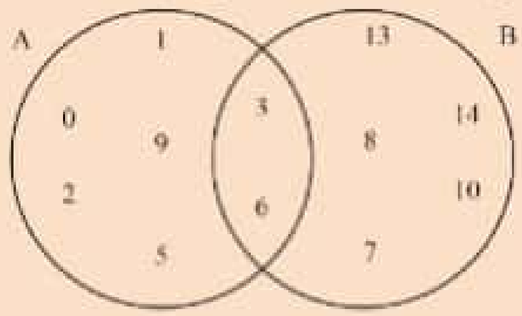
(a) $A' \cup B$ (b) $B' \cap A'$

(c) $A \cap B$ (d) $U \cup (A' \cap B)$

18. In a group of 29 tourists from different countries, 17 went to Manyara National Park, 13 went to Mikumi National Park and 8 went to neither Mikumi nor Manyara National Park. How many tourists went to both places?

19. Show by a Venn diagram that $A' \cup A = U$. Shade the required region.

20. Consider the following figure and answer the questions that follow.



(a) List the members of set A

(b) List the members of set B

(c) Find $n(A \cup B)$

21. In a certain street with 200 houses, 170 houses have electricity and 145 houses have glass doors. How many houses have both electricity and glass doors? Assuming that each house has either a glass door or electricity or both.

22. In an interview at a railway station, 50 travelers reported that last month they had been to Tanga, 48 had been to Arusha and 36 had been to both Arusha and Tanga. How many travelers were interviewed?

Chapter Eleven

Statistics

Introduction

It is easy to understand information presented in form of illustrations. Illustrations may be a table, a diagram, a picture, a chart or a graph. These illustrations are widely used in books, newspapers, atlases, and reports to present data. Statistics is the branch of mathematics that deals with collecting, organizing, analyzing and interpreting data. In this chapter, you will learn about pictograms, bar charts, line graphs, pie charts, frequency distributions, class mark of a class interval, graphs of frequency distributions, histograms, frequency polygons, and cumulative frequency curves or ogives. The competencies developed in this chapter will enable you to apply statistics in real life situations in various ways. For example to organise and analyse information/data statistically such as test/examination scores, attendance, and rainfall records for drawing conclusions and make informed decisions.

Pictograms

Activity 11.1: Collecting and presenting data

In a group perform the following:

1. Collect and record the following information in your exercise book:
   (a) Sizes of your classmates shoes.
   (b) The favourite subject of each classmate.
   (c) Ages of your classmates.
   (d) Heights of your classmates.
2. How can you present these data in the best way?

A pictogram is a drawing having pictures or symbols and is used to present statistical information. For example, the following pictogram shows the number of people who reported at a village dispensary during a certain week for HIV testing.

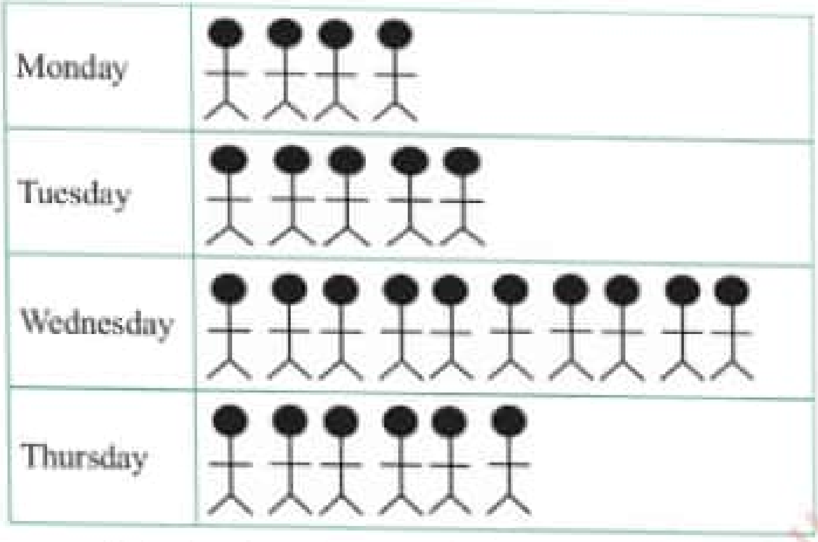
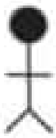


Figure 11.1: *Number of people who reported for HIV testing*

Key:  represents 10 people

The pictures (symbols) are always of the same size and the same distance apart, so that different rows can be compared easily. When the numbers of information items is large, one picture or symbol can represent 5, 10, 100, 1000 items, and so on.

Example 11.1

A town survey identified the following types of shops as given in the table below.

Types of items	Books	Food	Clothes	Household	Others
Number of items	5	15	20	25	10

Present this information by a pictogram.

Solution

Key: represents 5 shops

Exercise 11.1

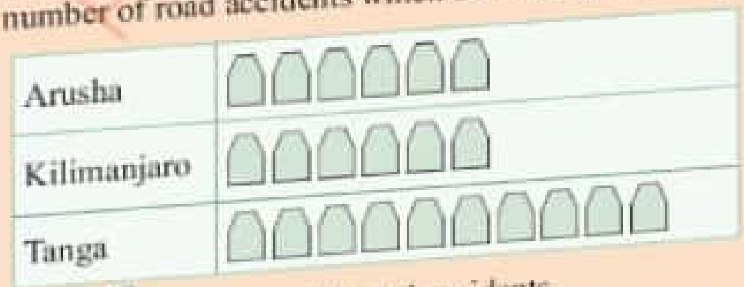
Answer the following questions:

1. The number of bags of potatoes received in Dar es Salaam markets from other regions in a certain week was as follows:

Region	Number of bags
Morogoro	40
Kilimanjaro	36
Iringa	44
Tanga	32
Mbeya	25

Present this information by a pictogram.

2. The following pictogram was published in a certain pamphlet to present the number of road accidents which occurred in three regions in 1969.



Key: represents 10 road accidents

(a) How many road accidents occurred in each region?

(b) Represent this information in a table.

3. The following table shows marks scored by students in a class of Form One in a Mathematics test.

Marks in percentage	20	30	45	50	55	60	65
Number of students	2	3	5	15	5	2	2

Use the table to answer the following questions:

(a) What is the lowest mark scored?

(b) What is the highest mark scored?

(c) Which mark is scored by the highest number of students?

(d) If 50% is the pass mark in the test, how many students passed the test?

(e) What is the difference between the highest and the lowest scores?

4. If one picture represents 1 000 people, draw a pictogram to present the following data showing the number of people infected with HIV in a certain region.

Year	Number of people
1998	30 000
1999	16 000
2000	12 000
2001	8000

Bar charts

A bar chart or bar graph presents data with rectangular bars with heights or lengths proportional to the values of data they represent. The bars can be vertical or horizontal, though they are usually vertical.

Before drawing a bar chart, a scale must be chosen. This is done by examining the data to find which item will be presented by the longest bar. Then, choose a scale so that the bar fits on a page.

Example 11.2

The following bar chart shows the amounts of monthly rainfall that were recorded in one year at a certain weather station.

Use the bar chart to answer the following questions:

(a) In which month was the highest amount of rainfall recorded?

(b) In which month was the lowest amount of rainfall recorded?

(c) How much rainfall was recorded in April?

(d) Which months had the same amount of rainfall?

Solution

(a) The highest amount of rainfall was recorded in May.

(b) The lowest amount of rainfall was recorded in August.

(c) The amount of rainfall recorded in April was 40 mm.

(d) The months that had the same amount of rainfall were January and June, March and December, July and September.

Exercise 11.2

Answer the following questions:

1. The number of people who were involved in road accidents on some days in a week were presented in the table as follows:

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Number of people	320	310	280	450	250

Draw a bar chart to present this information.

2. The following table shows the number of HIV patients in Tanzania from year 2000 to 2004.

Year	2000	2001	2002	2003	2004
Number of patients	82 000	98 900	113 200	127 800	147 300

Present this information using a bar chart.

3. The following table presents the number of students with their corresponding heights.

Height (cm)	130	135	136	138	140	141	144	145
Number of students	2	2	4	10	8	3	2	1

(a) How many students are 130 cm tall?
(b) What is the height of the tallest student?
(c) If it is decided that a basketball player should be at least 140 cm tall, how many students will qualify?

4. The following table shows the number of children who play various games on a given day. Present the data on a bar chart.

Games	**Number of children**
Rugby	20
Tennis	40
Hockey	35
Netball	10
Football	50
Squash	5

Line graphs

A line graph is a continuous line that joins points which represent data. In most cases, the data is concerned with variations over time. For example, in a certain hospital, the temperature of a patient of malaria taken after every 4 hours in a day was as follows:

Table 11.1: Malaria patient's temperature

Time	2:00 am	6:00 am	10:00 am	2:00 pm	6:00 pm	10:00 pm
Temperature (°C)	40.0	39.0	41.0	39.0	38.5	38.0

Figure 11.2: *A line graph of temperature against time*

Exercise 11.3

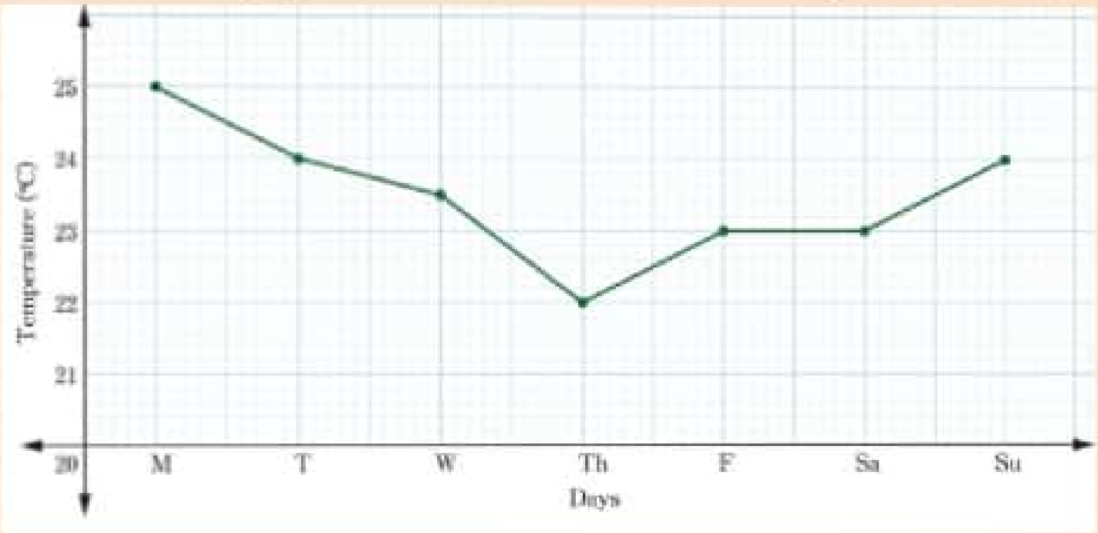
Answer the following questions:

1. The heights of a boy were measured and recorded on each birthday celebration in seven successive years as shown in the following table.

Age	10	11	12	13	14	15	16
Height (cm)	120	125	128	132	137	143	156

Draw a line graph to present the boy's heights.

2. The following figure shows temperatures at noon on the days of a certain week.



(a) Which was the hottest day of the week?

(b) Which was the coolest day of the week?

(c) What was the temperature on Tuesday?

3. The following figure is a line graph which shows the ages of a student and her corresponding masses.

(a) How old was she when the mass was 30 kilograms?

(b) How many kilograms did she gain from her 6^{th} year to her 11^{th} year of age?

(c) What would be her body mass at the age of 13 years?

4. The attendance of boys and girls in a certain school for five days in a week were as follows.

Day	Number of boys	Number of girls
1	425	450
2	400	445
3	420	430
4	415	435
5	390	410

Draw line graphs to present the attendance of both boys and girls on the same figure.

5. The following table shows the temperature of a patient in a certain hospital after every hour from 7:00 am to 11:00 am.

Time in hours	Temperature (°C)
7.00 am	38
8.00 am	38.7
9.00 am	38.9
10.00 am	37.7
11.00 am	37

(a) Present this information using a line graph.

(b) What was the highest temperature recorded?

(c) At what time was the patient's temperature normal?

Pie chart

Activity 11.2: Drawing a pie chart

1. Form a group of two or more students and prepare two pieces of plain papers, a pencil, a pair of compasses, and a protractor.
2. Draw a circle on each piece of paper using a pencil and a pair of compasses, mark the centres, and then construct an angle of 60° on one circle and 120° on the other.
3. What fraction of the circle is represented by each angle?

Another way of presenting data is by means of a pie chart, which is sometimes called a circle graph. Fractions of the total data are represented as slices of a 'pie', a round cake, which explains the name 'pie chart'.

Each fraction is multiplied by 360° to obtain the angle at the centre in degrees. A circle of convenient radius is drawn to represent the whole 'pie', then all angles at the centre are drawn using a protractor. Each sector of the circle is then labeled appropriately.

Note that, the numbers in a pie chart may be indicated either in percentages (%) or in degrees.

Example 11.3

A survey was done among 200 students to find the most popular subject. Each student voted for only one subject and the results were as follows:

Subjects	Math.	Eng.	Biol.	Hist.	Geog.	Phy.
Number of students	25	40	60	20	40	15

Present this information in a pie chart.

Solution

The angles representing each of the subjects are:

Mathematics: $\frac{25}{200} \times 360° = 45°$

English: $\frac{40}{200} \times 360° = 72°$

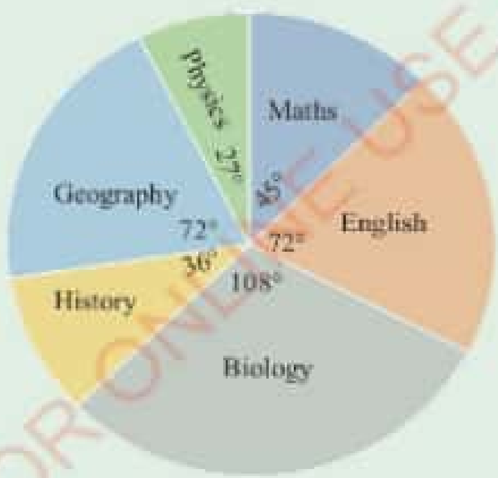
Biology: $\frac{60}{200} \times 360° = 108°$

History: $\frac{20}{200} \times 360° = 36°$

Geography: $\frac{40}{200} \times 360° = 72°$

Physics: $\frac{15}{200} \times 360° = 27°$

A pie chart for presenting the data from the survey.



Exercise 11.4

Answer the following questions:

1. In a group of 120 students, 30 prefer English, 40 prefer French, and 50 prefer Kiswahili. Present this information by a pie chart.

2. The following table shows the percentage of world environmental destruction in a certain year.

Continent	Europe	Africa	Asia	North America	Others
Percentage	27%	23%	20%	22%	8%

Present this information in a pie chart.

3. In a certain government report, it was reported that the exports of crops in a certain year were as follows:

Crop	Sisal	Cotton	Coffee	Pyrethrum	Others
Percentage	30	30	24	6	10

Draw a pie chart to present this information.

4. The following pie charts show the composition of exported materials by a certain country in the years 2012 and 2013.

2012:1 800 Million shillings

2013:2 400 Million shillings

(a) Which material had the largest percentage of the export market in:
   (i) 1973?   (ii) 1974?

(b) Estimate the value of sisal exported in:
   (i) 1973   (ii) 1974

Exercise 11.5

Answer the following questions:

1. In a certain country, the number of HIV patients in five regions were identified as follows:

Regions	A	B	C	D	E
HIV patients	120	80	260	140	200

Present this information by using a pie chart.

2. The following table shows the number of cars damaged in road accidents as were registered in one town in a certain year.

Types of cars	Scania	Datsun	Ford	Volkswagen
Number of damaged cars	4000	5000	1800	2000

Present this information by:

(a) A pictogram (b) A bar chart

3. The mass of an infant was measured for five consecutive months and the results were as follows.

Months	February	March	April	May	June
Mass (kg)	4.0	4.8	5.6	6.4	7.0

(a) Draw a line graph using this table of values.
(b) Estimate the infant's mass in mid-March.

4. The number of employees recorded absent on each day in a certain institution was as given in the following table:

Days	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.
Absentees	5	8	3	6	10	4

(a) Construct a bar graph to present the number of absentees on each day.
(b) Can this information be presented by using a line graph? Why?

5. The weights of A, B, C are in the ratio 2:3:4, respectively.
(a) Calculate the angles representing A, B and C.
(b) Present the angles on the pie chart.

6. The times to prepare meals L, M, and N are in the ratio $3:7:x$ On a pie chart, the angle corresponding to L is 60° Find the value of x.

Frequency distribution tables

Activity 11.3: Constructing a frequency distribution table

Steps:

1. Write all the letters of the English alphabet vertically on the blackboard/board.
2. Each student should mention his/her first name and make a tally on the board against a letter corresponding to first letter of her/his first name.
3. Find the total number of tallies for each alphabetical letter, and add them all.
4. Construct a table showing columns of alphabetical letters and the number of tallies for each letter.
5. Is the sum of tallies equal to the number of all students who participated in this activity?

A frequency distribution is a tallying of the number of times each data point or interval of data values occurs in a group of data. For example, let a Mathematics test be given to 10 students and let their scores be 8, 10, 8, 6, 6, 9, 5, 6, 4 and 7. We note that, there is one 10, two 8's, one 7, three 6's, one 4, one 5 and one 9. The frequency is the number of times a particular number occurs in a group. In this case, the frequency of 10 is 1 and that of 8 is 2, and so on. Assume a short Mathematics test was given to a class of 9 students, and that their scores were as follows: 6, 9, 10, 6, 4, 7, 5, 6 and 4. The first thing to do would be to arrange the scores in a descending or ascending order as shown in the following table.

Table 11.2: Scores on a mathematic test in descending order

Students	Scores
1	10
2	9
3	7
4	6
5	6
6	6
7	5
8	4
9	4

The next thing would be to find the frequency of each score by tallying as shown in following table. The table is called a frequency distribution table. The following table shows the frequency distribution of the scores of 9 students in Mathematics test.

Table 11.3: Frequency distribution table

Scores	Tally	Frequency
10	/	1
9	/	1
7	/	1
6	///	3
5	/	1
4	//	2
	Total	**9**

Usually, tallies are not shown in frequency distribution tables. They are helpful in getting the correct frequency. The number of observations is denoted by 'N' and the frequency by 'f'.

Example 11.4

Construct a frequency distribution table from the following data of ages of students in a certain class: 14, 15, 16, 14, 17, 15, 16, 13, 14, 14.

Solution

Frequency distribution table of ages of 10 students in a certain class.

Ages	Frequency
17	1
16	2
15	2
14	4
13	1
Total	**10**

When the given data is very large, it would be very difficult to present it using the previous method. The best way of presenting such data would be to group numbers by forming class intervals. A **class interval** is the number of data values which are grouped together. For example, 30 – 35 is a class interval of size 6. A class interval can be of any size depending on the size of population. The data from the previous table can be grouped as follows:

Table 11.4: Class intervals and their frequencies

Class interval	Frequency
16 – 17	3
14 – 15	6
12 – 13	1
Total	**10**

From class interval 14 – 15, the number 14 is called the **lower limit** and 15 the **upper limit** of the class interval. The boundary of the lower limit is 13.5 and that of the upper limit is 15.5. These are called **real limits** or **class boundaries.** The lower real limit is obtained by subtracting 0.5 from the lower limit. The upper real limit is obtained by adding 0.5 to the upper limit. Thus, for the class interval 12 – 13, the lower real limit is 11.5 and the upper real limit is 13.5.

Example 11.5

In a Mathematics test, the following marks were scored:

48 47 42 67 73 50 76 47 44 44 57 58 54 45

58 56 66 67 45 43 71 48 64 52 42 54 62 32

49 34 35 46 89 37 47 54 45 60 64 44

If the size of the class interval is 8, group the marks starting with the interval 32 – 39, hence draw the frequency distribution table.

Solution

With the help of tallies, the frequency distribution table is as shown in the following table.

Class interval	Tally	Frequency
32 – 39	////	4
40 – 47	~~////~~ ~~////~~ ///	13
48 – 55	~~////~~ ///	8
56 – 63	~~////~~ /	6
64 – 71	~~////~~ /	6
72 – 79	//	2
80 – 87		0
88 – 95	/	1
Total		**40**

The class size of a class interval is the difference between the upper real limit and lower real limit.

Example 11.6

The following table shows the frequency distribution of scores in an English test. Calculate the size of the class interval.

Class interval	Frequency
50 – 59	2
40 – 49	4
30 – 39	6
20 – 29	3
10 – 19	2
	N = 17

Solution

Taking any class interval, say 30-39, the size of this class interval is given by 39.5 – 29.5 =10. Therefore, the size of the class interval is 10 and is the same for all class intervals.

Class mark of a class interval

The class mark of a class interval is the central value of the class interval. It is the value which is half way between the class limits. It is sometimes called the class midpoint. For example, the class mark for the class interval 30 – 39 is the average of the lower limit and the upper limit.

$$\text{Class mark} = \frac{\text{Upper class limit} + \text{Lower class limit}}{2}$$

In this case, it is $\frac{30+39}{2} = 34.5$.

Example 11.7

The following table shows the frequency distribution in a Mathematics test.

Class interval	Frequency
81 – 90	1
71 – 80	3
61 – 70	5
51 – 60	10
41 – 50	17
31 – 40	4

Find the following:

(a) The size of the class interval.

(b) The class marks for the intervals 41 – 50 and 81 – 90.

(c) The number of candidates who sat for the test.

Solution

The size of the class interval is 40.5 – 30.5 = 10

The class mark for the interval 41 – 50 is $\frac{41+50}{2} = 45.5$ and the class mark for the interval 81 – 90 is $\frac{81+90}{2} = 85.5$.

The number of candidates is determined by adding all the frequencies.

Thus, $4 + 17 + 10 + 5 + 3 + 1 = 40$

Therefore, the number of candidates who sat for the test was 40.

Exercise 11.6

Answer the following questions:

1. In a Biology test, the following marks were scored:

54	54	40	55	54	43	73	37	75	47
35	47	73	46	31	43	47	35	35	60
69	51	44	48	55	45	50	37	51	36

Group the marks in class intervals of 20 – 29, 30 – 39, 40 – 49, …, and then construct the frequency distribution table.

2. In a certain clinic, the number of children delivered in 20 days was recorded as presented in the following table:

2	5	3	4	2	4	6	4	8	3
5	1	6	8	3	3	7	4	2	3

Construct a frequency distribution table for this data.

3. The following is a table of the masses of 36 men in kilograms:

51	61	60	70	75	71	75	70	74	73	72	82
83	83	80	50	55	62	62	61	70	71	76	74
50	68	68	66	65	72	69	64	63	58	90	89

Present the masses in a frequency distribution table with class intervals of class size 5, starting with the class interval 50 – 54.

4. In a Mathematics examination, the following marks were scored:

27	57	57	40	70	48	59	60	42	44
47	44	44	59	35	48	43	52	36	48

Group the marks in class interval of 20 – 29, 30 – 39, 40 – 49, . . . and so on and then construct the frequency distribution table.

5. The following is a set of marks in a Geography examination. Construct the frequency distribution table with class intervals, real limits, class marks, and interval size, starting with the class interval 8 – 15.

54	81	18	44	24	63	67	60	34	39
91	47	75	72	36	87	49	86	57	74
26	41	90	59	14	13	31	68	13	29
29	70	22	63	35	50	42	27	95	77
42	31	69	73	11	31	45	51	56	40

6. In an experiment, 50 students were asked to guess the weight of a bunch of bananas in kilograms. The responses were as follows:

47	39	21	30	42	35	44	36	19	52
23	32	66	29	5	40	33	11	44	22
27	58	38	37	48	63	23	40	53	24
47	22	44	33	13	59	33	49	57	30
17	45	38	33	25	40	51	56	28	64

(a) Construct a frequency distribution table using class intervals 0 – 9, 10 – 19, 20 – 29, and so on.

(b) Find the class marks.

Histograms

Activity 11.4: Drawing a histogram of a frequency distribution.

In groups of ten members do the following:

1. Ask your fellow students to give their heights.
2. List the heights of the students.
3. Group the heights in class intervals of size 5 and construct a frequency distribution table.
   (a) Draw y and x axes to represent frequency and height respectively, on graph paper.
   (b) Draw rectangular bars representing height against frequency.
   (c) Shade the drawn bars.

A histogram or a frequency distribution is made up of rectangular bars with bases on the horizontal axis and centres at the class marks. It has a width equal to the class interval size. The heights of the rectangles are proportional to the class frequencies. In fact, their areas are proportional to the class frequencies. Suppose we want to draw a histogram of 100 Mathematics examination scores as given in Table 11.5.

Table 11.5: Mathematics examination scores by class intervals, class marks and frequency.

Class interval	Class marks	Frequency
95 – 99	97	3
90 – 94	92	7
85 – 89	87	9
80 – 84	82	13
75 – 79	77	20
70 – 74	72	23
65 – 69	67	17
60 – 64	62	8

Use a graph paper to draw the rectangles using frequencies as heights and class intervals as widths for class marks. For example, draw a rectangle of height 8 and corresponding to the frequency 8 and an interval with class mark 62. This can be done for all the intervals corresponding to the given frequencies. When all rectangles have been completed, the histogram will be as shown in the following figure.

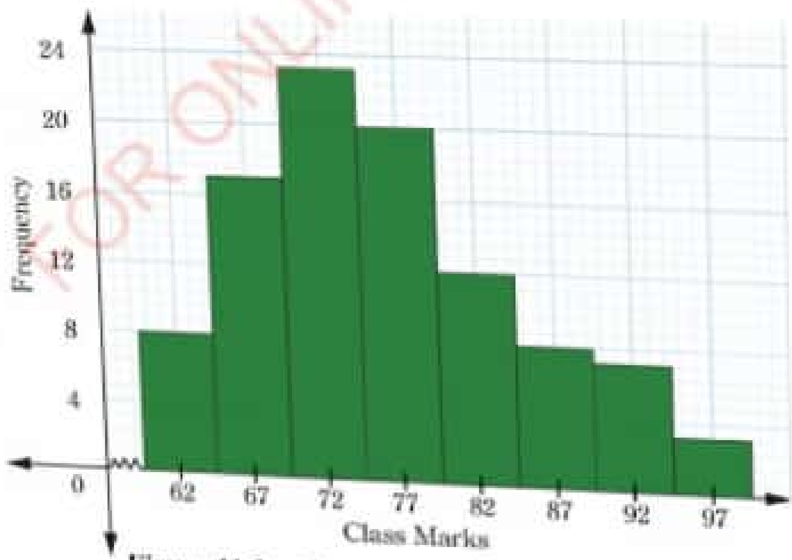


Figure 11.3: *A histogram or frequency histogram*

Usually, frequencies are plotted against class marks. The frequencies are represented on the y-axis, and class marks on the x-axis. Appropriate scales must be chosen on both axes.

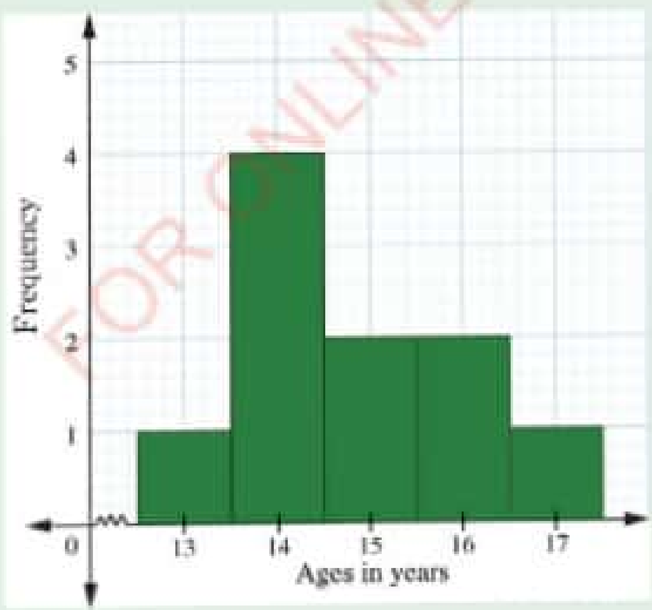
Example 11.8

The following frequency distribution table represents the ages of 10 students in years. Represent the information using a histogram.

Age	Frequency
17	1
16	2
15	2
14	4
13	1

Solution

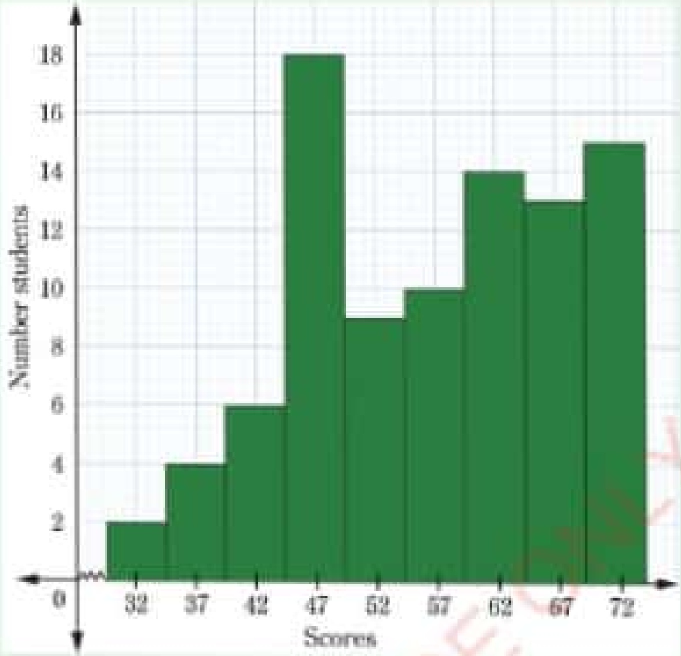
Taking one unit on the vertical line to represent one unit of frequency and one unit on the horizontal line to represent one unit of age, we obtain the histogram as shown in the following figure.



Math F2 – (BOOK) (DUMMY) Sept 4.indb 351

Example 11.9

The following figure is a histogram for an English test scores of 91 students. Use this figure to answer the questions that follow:



(a) How many students scored marks between 30 and 34?

(b) Which class interval had the highest number of students?

(c) What is the frequency for the class mark 67?

(d) What is the size of the class interval for this distribution?

Solution

(a) The class interval 30 – 34 has frequency 2. Therefore, 2 students got scores between 30 and 34.

(b) The class interval 45 – 49 with class mark 47 has the highest frequency which is 18.

(c) The class mark 67 has frequency 13.

(d) The size of the class interval can be found by using the class marks. Take class mark 42, add it to the next upper class mark, that is 47, and then divide the sum by 2.

Thus, $\frac{42+47}{2}=44.5$. Adding 42 to the previous lower class mark, and then divide by 2. Results to, $\frac{42+37}{2}=39.5$.

The class size is then given by upper real limit minus lower real limit, that is $44.5-39.5=5$.

Therefore, the size of the class interval for the distribution is 5.

Frequency polygons

Activity 11.5: Drawing a frequency polygon

1. In pairs or groups plot the following points in the xy – plane and join them using a straight edge.
   (a) A (1, 0), B (4, 2), C (7, 3), D (9, 1) and E (12, 0)
   (b) P(3, 4), Q1, 5), R(–1, 4) and S(1, 3)
2. What is the name of the figure formed in each case?

A frequency polygon is a line graph of class frequencies plotted against class marks. When drawing a frequency polygon, two extra class marks are introduced, one to the left most class mark and the other to the right most class mark. The new class marks introduced are assigned zero frequencies. The points are then placed directly over the class marks of each class interval. The adjacent points including zero are joined by straight lines as shown in Figure 11.4.

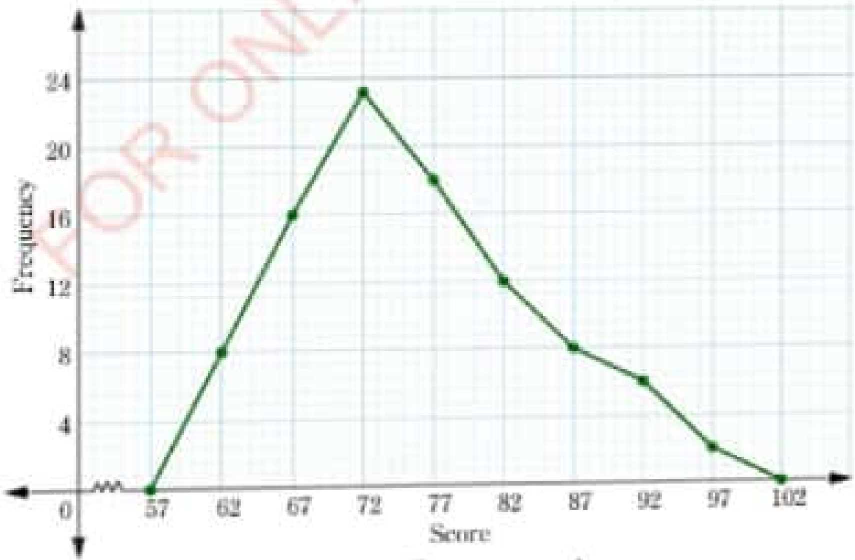


Figure 11.4: *A Frequency polygon*

Math F2 – (BOOK) (DUMMY) Sept 4.indb 353

Example 11.10

The following data represents the time, in minutes, taken by competitors to complete a certain task. Draw a frequency polygon to present this information.

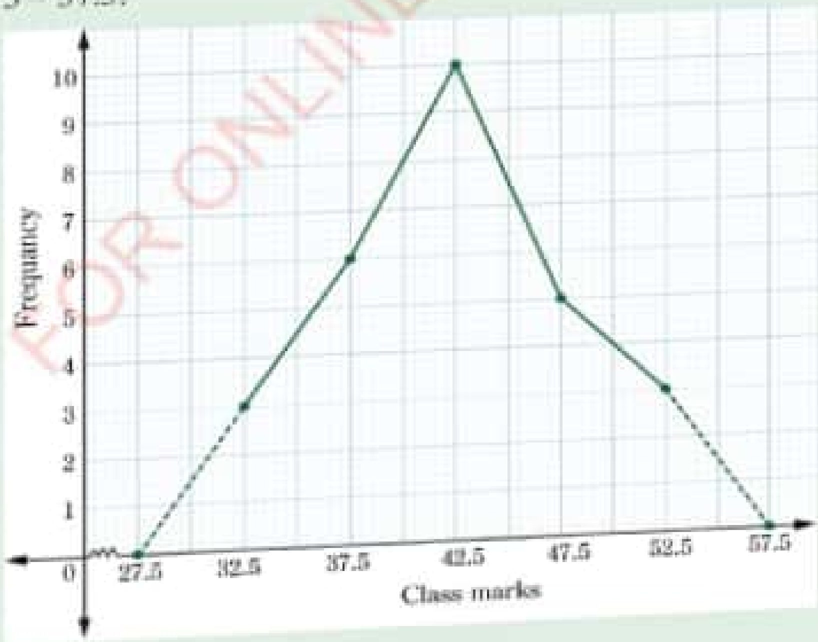
Time (minutes)	30 – 35	35 – 40	40 – 45	45 – 50	50 – 55
Frequency	3	6	10	5	3

Solution:

Frequency polygon of the information.

Time (minutes)	Frequency (f)	Class marks
30 – 35	3	32.5
35 – 40	6	37.5
40 – 45	10	42.5
45 – 50	5	47.5
50 – 55	3	52.5
	N = 27	

Introduce one class mark to the left of 32.5 with frequency 0 and one class mark to the right of 52.5 with frequency 0. Since the class size is 5, then the class mark to the left of 32 will be 32.5 – 5 = 27.5 and the class mark to the right of 52.5 will be 52.5 + 5 = 57.5.



Example 11.11

1. The following table shows the ranges of salaries of 91 employees in a certain hospital:

Salary (Tshs)	Number of employees
120 000 – 210 000	6
220 000 – 310 000	9
320 000 – 410 000	20
420 000 – 510 000	15
520 000 – 610 000	10
620 000 – 710 000	18
720 000 – 810 000	13

(a) Construct a frequency distribution table for this data.

(b) Draw a frequency polygon to present the data.

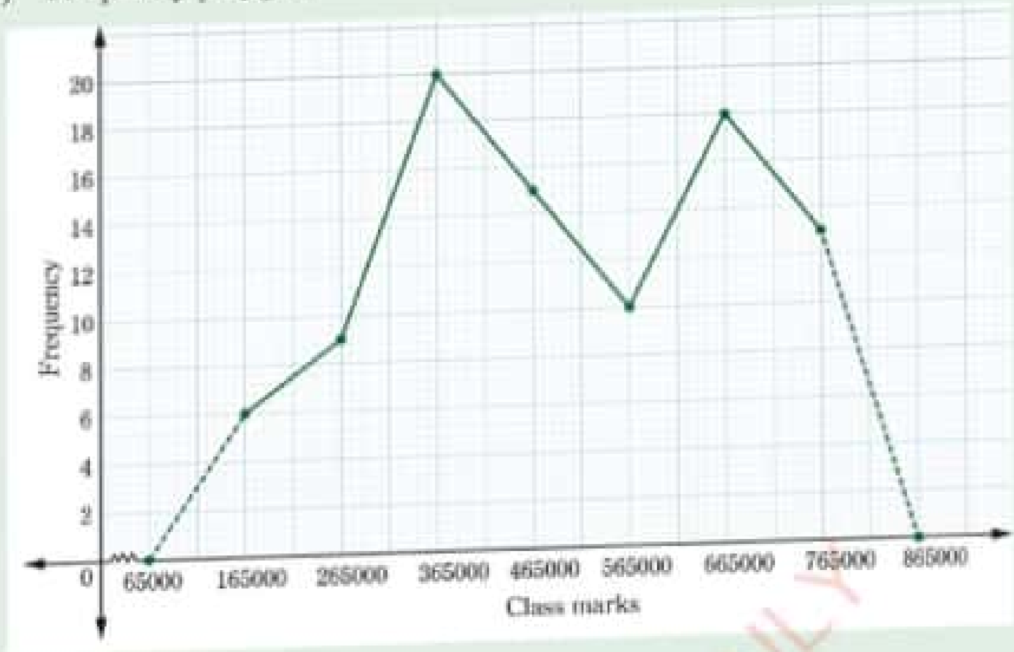
(c) What is the class size of the class intervals?

Solution

(a) Frequency distribution table.

Salary (Tshs)	Number of workers (y)	Class mark (x)
120 000 – 210 000	6	165 000
220 000 – 310 000	9	265 000
320 000 – 410 000	20	365 000
420 000 – 510 000	15	465 000
520 000 – 610 000	10	565 000
620 000 – 710 000	18	665 000
720 000 – 810 000	13	765 000
	N = 91	

(b) Frequency polygon



(c) The difference between the lower limit of the second class interval and the upper limit of the first class interval is given by:

$$220\,000 - 210\,000 = 10\,000$$

Then, $\frac{10\,000}{2} = 5\,000$

From the class interval 120 000 – 210 000

The class size is found by taking the upper class boundary minus lower class boundary.

Thus, the upper class boundary will be 210 000 + 5000 = 215 000.

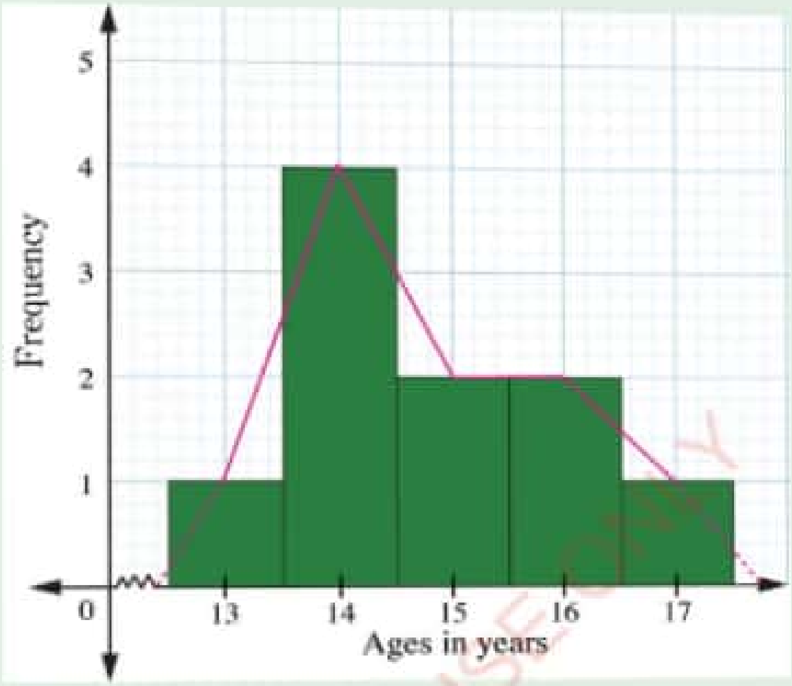
The lower class boundary will be 120 000 – 5000 = 115 000.

Hence,

 215 000 (upper class boundary)
–115 000 (lower class boundary)
 100 000 (class size)

Therefore, the class size is 100 000.

Note that, a frequency polygon can also be drawn by joining the mid – points of the tops of the bars of the corresponding histogram. The following figure shows a frequency polygon and a histogram plotted on the same graph.



Exercise 11.7

Answer the following questions:

1. The following table shows female death risks between 0 and 39 years. Present this information using:
   (a) A histogram
   (b) A frequency polygon.

Expected death risk of females per 1000 women	
Age in years	**Female death risks**
35 – 39	125
30 – 34	120
25 – 29	110
20 – 24	95
15 – 19	60
10 – 14	55
5 – 9	95
0 – 4	340
	N = 1000

2. The following tables show the distribution of marks scored by 40 students in two different monthly tests. Draw their frequency polygons on the same graph to present this information.

Test A	
Marks	**Frequency**
71 – 80	1
61 – 70	3
51 – 60	5
41 – 50	10
31 – 40	17
21 – 30	4
	N = 40

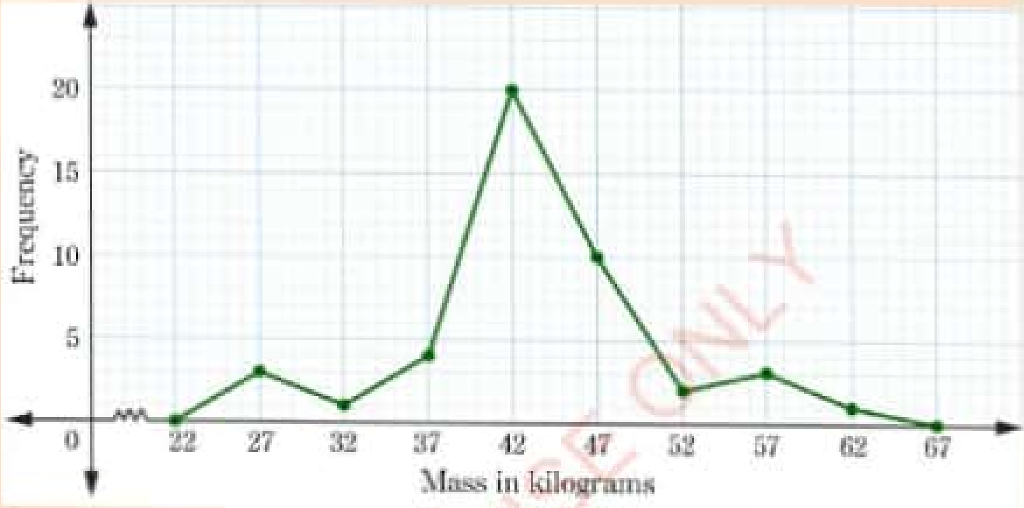
Test B	
Marks	**Frequency**
71 – 80	2
61 – 70	12
51 – 60	15
41 – 50	4
31 – 40	3
21 – 30	4
	N = 40

3. The heights, in centimetres, of 10 students in a certain class were recorded as follows:

Height (cm)	130	135	140	145	150
Frequency	3	1	3	1	2

Present this data using a frequency polygon.

4. The following figure is a frequency polygon of masses of students in kilograms.



Use this frequency polygon to answer the following questions:

(a) How many students were there?

(b) What is the size of the class interval?

(c) What is the lowest class interval?

(d) How many students had mass of 42 kilograms?

5. The following frequency distribution table shows the distance (in km) traveled by students from their homes to school.

Distance (km)	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Number of students	2	6	8	6	3

Draw a frequency polygon to present this information.

6. Can every data set be displayed using a histogram? If yes, explain why? If no, give a counter example and explain why not.

7. Determine whether the statement is **True** or **False:**

(a) The vertical scale for a histogram must be based on equal intervals.

(b) The axes of a histogram are not needed.

Cumulative frequency curves or Ogive

Frequency tables show how often individual data items occur in a frequency distribution. For example, a frequency distribution of marks scored by students in a test shows how many students scored a particular mark. However, sometimes we are not interested in the frequency of occurrence of an individual mark, but rather, how many students scored less than a certain mark. To answer this question we use the concepts of cumulative frequency distribution and cumulative frequency curves or ogives.

Example

The following table shows the number of students who scored marks in various class intervals in an examination.

Marks	0 – 19	20 – 39	40 – 59	60 – 79	80 – 100
Frequency	7	21	38	27	7

The cumulative frequencies table for this data is:

Cumulative frequency	7	28	66	93	100

Tabulation of the cumulative frequencies against the corresponding upper class boundaries leads to the following cumulative frequency distribution.

Upper class boundary	19.5	39.5	59.5	79.5	100.5
Cumulative frequency	7	28	66	93	100

This distribution shows how many students scored marks less than each upper class boundary. For example, it shows that 66 students scored less than 59.5 marks.
The smooth curve passing through all points obtained by plotting the cumulative frequencies against the corresponding upper class boundaries is called a cumulative frequency curve or an ogive.

Activity 11.6: Constructing a cumulative frequency table of students' weights in a class.

In pairs or a group of 10 students perform the following:

1. Use a weighing balance to measure the weight of each student in the group and take records.
2. Construct a cumulative frequency table of the weights with a class size of 2 kilograms.
3. Discuss how you can use the cumulative frequency table to draw a cumulative frequency curve.

Example 11.12

The following table is a frequency distribution of English test scores for 90 students.

Scores	Frequency
70 – 74	16
65 – 69	12
60 – 64	14
55 – 59	10
50 – 54	8
45 – 49	18
40 – 44	6
35 – 39	4
30 – 34	2
	N = 90

(a) Construct a cumulative frequency distribution.

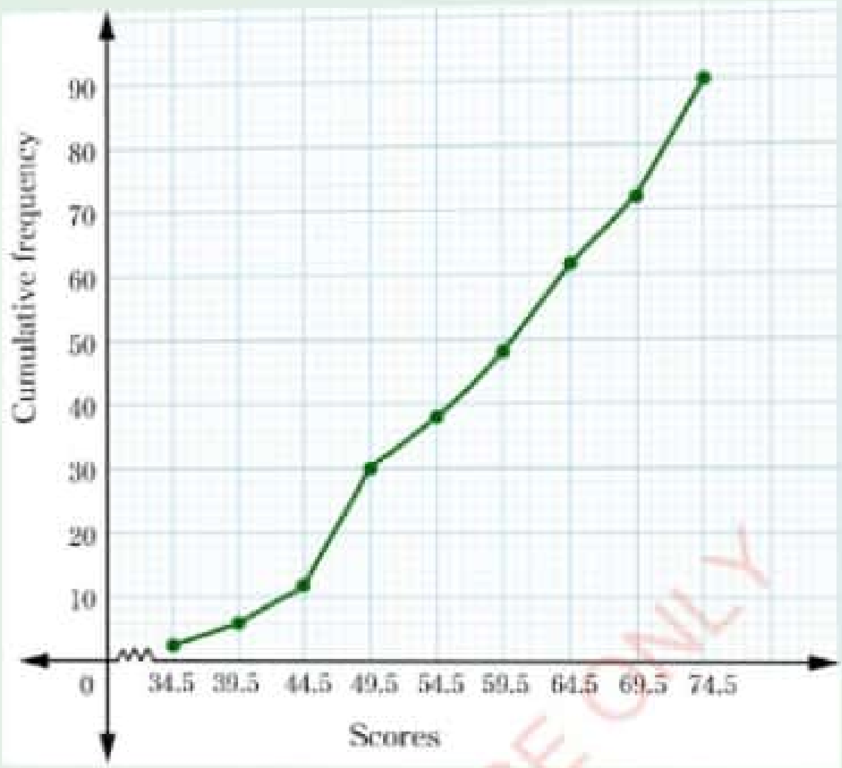
(b) Draw a cumulative frequency curve or ogive.

Solution

(a) For each class interval, find its upper real limit and add the frequencies as shown in the following table.

Scores	Class interval Frequency	Cumulative frequency
less than 34.5	2	2
less than 39.5	4	6
less than 44.5	6	12
less than 49.5	18	30
less than 54.5	8	38
less than 59.5	10	48
less than 64.5	14	62
less than 69.5	12	74
less than 74.5	16	90

(b) The cumulative frequency curve is as shown in the following figure.



Example 11.13

A motor vehicle manufacturing company tested 100 cars to see how far they could travel using 10 litres of petrol. The results were recorded as shown below:

Distance (km)	100 – 109	110 – 119	120 – 129	130 – 139	140 – 149
Number of cars	5	15	25	35	20

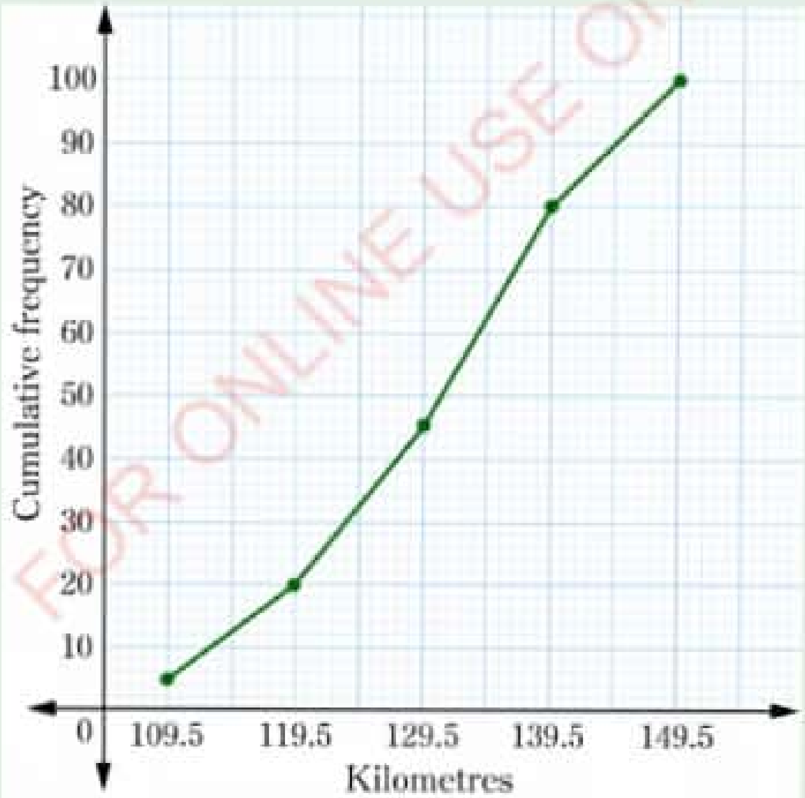
Draw a cumulative frequency curve for presenting this information.

Solution

Construct a cumulative frequency table as shown in the following table.

Distance (km)	Frequency	Cumulative frequency
less than 109.5	5	5
less than 119.5	15	20
less than 129.5	25	45
less than 139.5	35	80
less than 149.5	20	100

Cumulative frequency curve for the distribution.



Math F2 - (BOOK) (DUMMY) Sept 4.indb 363

Example 11.14

The ages (in years) of ten students were recorded as follows:

13 years – 1 student

14 years – 4 students

15 years – 2 students

16 years – 2 students

17 years – 1 student

Use this information to construct a cumulative frequency table.

Solution

Age	Frequency	Cumulative frequency
Less than 13.5	1	1
Less than 14.5	4	5
Less than 15.5	2	7
Less than 16.5	2	9
Less than 17.5	1	10

Example 11.15

Using the Ogive drawn in example 11.13, answer the following questions:

(a) How far would the 25[th] car have travelled?

(b) How many cars would have travelled 124.5 kilometres?

Solution

Using the ogive drawn in Example 11.13.

(a) On the cumulative frequencies axis, take the point halfway between 20 and 30. From the point move horizontally until the ogive curve is met. Then, move down along the vertical line until the axis representing distance is met. It can be observed that the vertical line meet at 121 kilometres with the distance axis. Therefore the 25[th] car travelled 121 kilometres.

(b) On the axis representing distances, locate a point 124.5, from this point move up until the orgive curve is met. From the orgive curve move to the left until the axis representing cumulative frequencies is met. It can be observed that the cumulative frequency is met at 45. Therefore, 45 cars travelled 124.5 kilometres.

Exercise 11.8

Answer the following questions:

1. In the following table, complete the cumulative frequency column:

Number of beds	Frequency	Cumulative frequency
70 – 79	5	
60 – 69	5	
50 – 59	1	
40 – 49	12	
30 – 39	12	
20 – 29	15	

2. The following table gives the frequency distribution of marks scored by 130 students in Biology and History subjects.

Percentage	Biology		History	
	Frequency	Cumulative frequency	Frequency	Cumulative frequency
91 – 100	13	130	1	130
81 – 90	22	117	1	129
71 – 80	31	95	0	128
61 – 70	30	64	5	128
51 – 60	24	34	9	123
41 – 50	6	10	25	114
31 – 40	3	4	28	89
21 – 30	1	1	30	61
11 – 20	0	0	26	31
1 – 10	0	0	5	5

(a) Draw their ogives on one diagram.

(b) If the pass mark was 51 use the distribution to find the percentage of students who failed in each of the subjects.

3. A survey was made on 100 football matches. The total number of goals was recorded and a frequency table was constructed as follow:

Goals	Frequency	Cumulative frequency
11	1	
10	0	
9	0	
8	2	
7	5	
6	7	
5	10	
4	16	
3	22	
2	17	
1	15	
0	5	

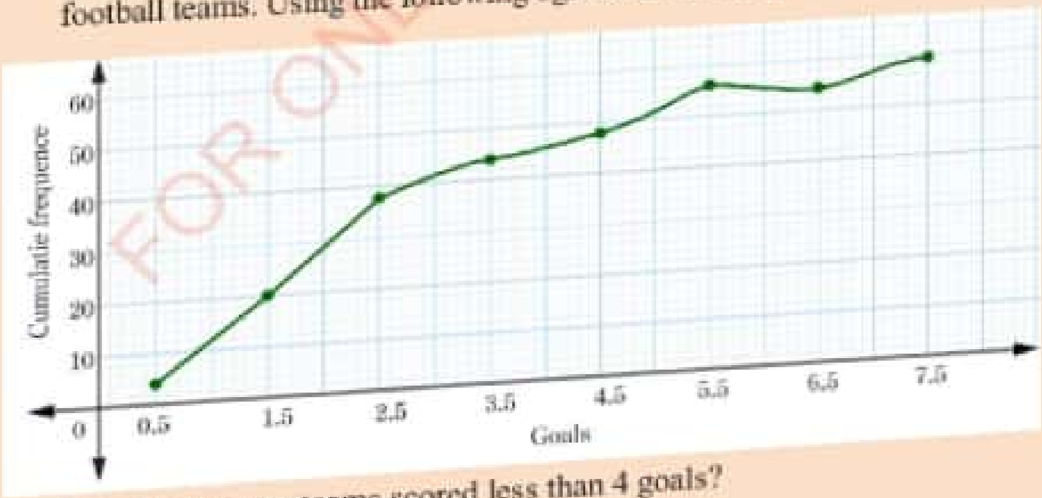
(a) Complete the cumulative frequency column.

(b) How many matches recorded less than 7 goals?

(c) How many matches recorded at least 6 goals?

(d) How many matches recorded between 4 and 7 goals?

4. The cumulative frequency graph below shows goals scored in one day by 54 football teams. Using the following ogive answer the questions that follows:



(a) How many teams scored less than 4 goals?

(b) How many teams scored less than 7 goals?

5. In a competition, 30 children had to pick up as many paper clips as possible in one minute. The results were as follows:

3	17	8	11	26	23	18	28	33	38
12	38	22	50	5	35	39	30	31	43
27	34	9	25	39	14	27	16	33	49

Construct a frequency distribution table using intervals 1 – 10, 11– 20, 21 – 30 and so on. Hence, draw a cumulative frequency curve.

Chapter summary

1. Statisticsis the branch of mathematics that deals with collecting, organizing, representing and interpreting numerical data in large quantities especially for the purpose of inferring proportions in a whole from those in a representative sample.
2. Frequency is the number of times an event or data value occurs.
3. Pictogram is a method of presenting statistical information using pictures or symbols.
4. Bar charts are long rectangles whose lengths or heights are proportional to the data.
5. A line graph is a continuous line that joins points which represent data.
6. A pie chart is a method of presenting data in a circular form, where the slices of a pie show the relative size of the data.
7. A histogram is a block graph that presents grouped distributions. In a histogram, the areas of the blocks or rectangles represent the frequency distribution. The horizontal width of each rectangle corresponds to the class width.
8. A frequency polygon is a line graph that presents grouped data graphically.
9. A cumulative frequency curve or ogive, is a curve that shows the relationship between cumulative frequency and corresponding data.
10. Class mark is the central value of a class interval, it is the value which is half way between the class limits.

Revision exercise 11

Answer the following questions:

1. The ages of 22 players in a football match were recorded in the following array:

   17, 18, 15, 16, 16, 16, 15, 15, 18, 15, 15,
   18, 18, 15, 16, 17, 15, 15, 16, 15, 18, 15.

   Present the data in a frequency table.

2. The following table is the frequency distribution of the masses of 45 students in a class.

Mass in kilograms	41 – 45	46 – 50	51 – 55	56 – 60	61 – 65	66 – 70
Number of students	3	8	13	11	7	3

   Draw a histogram of the distribution.

3. The examination marks of 45 students were as follows:

65	58	71	62	64	35	72	32	64
46	59	82	73	76	54	63	63	75
71	61	36	64	80	61	64	76	64
60	68	48	35	92	73	46	24	35
43	30	50	70	40	46	64	27	28

   (a) Construct a frequency distribution using class intervals 21 – 30, 31 – 40, 41 – 50, and so on.

   (b) Draw a cumulative frequency curve.

4. A tailors' association surveyed the number of sewing machines that each primary association had and recorded as follows:

   3, 8, 5, 4, 5, 5, 6, 7, 7, 4, 6, 6, 7, 4, 4, 6, 7, 5, 4, 3, 7, 5, 7, 8, 3.

   Construct a frequency table, and then draw a frequency polygon.

5. In a certain examination, the results were as follows:

3 students scored marks between 0 and 10,

5 students scored marks between 10 and 20,

5 students scored marks between 20 and 30,

4 students scored marks between 30 and 40,

2 students scored marks between 40 and 50.

Draw a histogram to present the examination results.

6. The payment of 36 labourers in Tanzanian shillings were given in a cumulative frequency polygon as shown in the following figure.

(a) How many labourers had salaries of 1200 shillings?

(b) What is the payment of the first 15 labourers?

7. The annual scores of Form Two students in a History examination were recorded as shown in the following table:

Scores	Frequency
95 – 99	4
90 – 94	9
85 – 89	6
80 – 84	21
75 – 79	12
70 – 74	13
65 – 69	10
60 – 64	11
55 –59	2
50 – 54	1

(a) What is the size of the class intervals?

(b) Draw a histogram to present the examination results.

Answers to odd – numbered questions

CHAPTER 1

Exercise 1.1

1. (a) Base 6, exponent 17
   (b) Base 7, exponent 4
   (c) Base −10, exponent 19
   (d) Base 3, exponent 800
   (e) Base 6, exponent 2
   (f) Base 50, exponent 0
   (g) Base y, exponent 25
   (h) Base 17, exponent 6
   (i) Base $\frac{5}{6}$ exponent 9
   (j) Base $\frac{1}{2}$, exponent 17
   (k) Base 19, exponent 101
   (l) Base −75, exponent 8
   (m) Base $x + y$, exponent n
   (n) Base $7 + x$, exponent n

3. (a) 7^4 : base 7, exponent 4
   (b) $(-2)^7$: base −2, exponent 7
   (c) 14^3 : base 14, exponent 3
   (d) 19^1: base 19, exponent 1
   (e) $(x + b)^4$ base $(x + b)$, exponent 4
   (f) $(-r)^7$: base $(-r)$, exponent 7
   (g) 50^{30} : base 50, exponent 30
   (h) 5^9: base 5, exponent 9
   (i) $(a + b)^2$: base $(a + b)$, exponent 2
   (j) $\left(\frac{w}{8}\right)^6$: base $\frac{w}{8}$, exponent 6
   (k) $(0.3)^5$: base 0.3, exponent 5
   (l) v^3 : base v, exponent 3

5. (a) 5^3 (b) 6^2 (c) 12^3 (d) 2^4 (e) 10^{12}

Exercise 1.2

1. (a) 10^4 (b) $\left(\frac{1}{2}\right)^8$ (c) r^{12} (d) $\left(\frac{7}{6}\right)^{14}$
   (e) $12a^6b^8$ (f) $\left(\frac{17}{20}\right)^6$ (g) 16^{10} (h) $\left(\frac{3}{4}\right)^5$
   (i) 10^7 (j) $(0.58)^{24}$ (k) 3^{32} (l) 10^{p+r+10}

3. (a) $4^2 \times 3^2$ (b) $2^2 \times a^2$ (c) $3^2 \times 7^2$
(d) $2^2 \times x^4$ (e) $2^3 \times t^6$ (f) $10^3 p^6$
(g) $5 \times m^2 \times n^2$ (h) $3 \times a^6 \times b^3$ (i) $2^2 \times a^6 \times b^8$
(j) $7^{69} \times x^{69}$

Exercise 1.3

1. x^1 3. 4 5. 33
7. x^4 9. 256 or 2^8 11. 10 000 or 10^4
13. $m^2 n$ 15. $4a^2$ 17. $2x^{-3}$ or $\frac{2}{x^3}$
19. $3bc$ 21. $6a^2b^2$ 23. $8a$
25. $54y$ 27. $3^4 \times 2^{13} \times 5^2$ 29. $17^2 \times 2^6$

Exercise 1.4

1. $\frac{1}{x^3}$ 3. $\frac{1}{x^{10}}$ 5. $\frac{r^7}{7^7}$ 7. $\frac{1}{3}$ 9. 6^7

11. 2 13. $\frac{1}{p^{23}}$ 15. $\frac{1}{a^3}$ 17. $\frac{1}{a^8}$ 19. $\frac{1}{y^4}$

21. m^{10} 23. $\frac{1}{p^9}$ 25. $\frac{1}{x^9}$ 27. $\frac{b^{17}}{a^{12}}$ 29. $\frac{3h^3}{r^2}$

Exercise 1.5

1. $a = 2$ 3. $x = 2$ 5. $x = 4$ 7. $x = 4$
9. $y = 2$ 11. $x = 6$ 13. $y = 1$ 15. $x = 12$

Exercise 1.6

1. 8   3. 10   5. $\frac{1}{2}$ 7. $\frac{1}{10}$ 9. $\frac{3}{10}$

11. 3   13. $\frac{11}{10}$ 15. $\frac{1}{10}$ 17. $\frac{3}{10}$

Exercise 1.7

1. (a) 13   (b) 27   (c) $32\sqrt{2}$ (d) 50   (e) 32
   (f) 8   (g) 7   (h) 10   (i) 9

3. (a) $\sqrt{50}$ (b) $\sqrt{176}$ (c) $\sqrt{90}$ (d) $\sqrt{243}$ (e) $\sqrt[3]{250}$
   (f) $\sqrt[3]{81}$ (g) $\sqrt[3]{-8\,000}$ (h) $\sqrt[3]{864}$ (i) $\sqrt[3]{1715}$

Exercise 1.8

1. $8\sqrt{3}$ 3. $8\sqrt{30}$ 5. $2\sqrt{5}$ 7. $5\sqrt{5}$

9. $9\sqrt{2}-6\sqrt{6}$ 11. $4\sqrt{2}$ 13. $9\sqrt{2}-6\sqrt{3}$ 15. $4\sqrt{2}$

17. $-\frac{\sqrt{7}}{2}$ 19. $26\sqrt{6}$

Exercise 1.9

1. 10   3. $9\sqrt{30}$ 5. 90   7. 60

9. $6\sqrt{6}$ 11. $65\sqrt{5}$ 13. 40   15. $40\sqrt{3}$

17. $4\sqrt{10}$ 19. $31+5\sqrt{5}$ 21. 2   23. $13+4\sqrt{3}$

Exercise 1.10

1. 5   3. $\frac{\sqrt{6}}{5}$ 5. $5\sqrt{3}$ 7. $\frac{2\sqrt{3}}{3}$

9. $\frac{\sqrt{3}}{2}$ 11. $\frac{2}{3}$ 13. $\frac{2\sqrt{5}}{3}$ 14. 6

Exercise 1.11

1. $\frac{\sqrt{5}}{5}$
3. $\frac{\sqrt{6}}{6}$
5. $\frac{5-\sqrt{10}}{15}$
7. $\frac{\sqrt{21}+8\sqrt{3}}{3}$
9. $4+2\sqrt{6}$
11. $5\sqrt{7}+5\sqrt{6}-\sqrt{42}-6$
13. $\frac{-9\sqrt{15}+27\sqrt{6}+\sqrt{10}-6}{13}$
15. $\frac{(x-2y)(\sqrt{x}-\sqrt{y})}{x-y}$
17. $\frac{81+29\sqrt{5}}{88}$
19. $\frac{8\sqrt{70}+56+3\sqrt{3}-3\sqrt{7}}{12}$

Exercise 1.12

1. 46.43
3. 16.34
5. 0.75
7. 0.0775
9. 160.25
11. 25.42
13. 12.41
15. 1.2

Exercise 1.13

1. (a) 2.6684 (b) 4.0000 (c) 3.6401 (d) 1.8443
3. 1.1874

Exercise 1.14

1. $R=\frac{100I}{PT}$
3. $h=\frac{2A}{b}$
5. $P=\frac{100A}{100+TR}$
7. $\frac{4T^2}{9}$
9. $t=\sqrt{\frac{2s}{a}}$
11. $a=\frac{p-w}{p+w}$
13. $F=\frac{9}{5}C+32$

Revision exercise 1

1. (a) 30 (b) 400 (c) 30

3. (a) $20\sqrt{3}$ (b) 34 (c) 10 (d) $4\sqrt{7}$
   (e) $\frac{5}{2}\sqrt{\frac{1}{2}}$ (f) $8\sqrt{10}-3\sqrt{7}$ (g) $5\sqrt{6}$ (h) 15

5. (a) $2y$ (b) $2m\sqrt{2ym}$ (c) $2y\sqrt{6y}$ (d) $y\sqrt{x}$
   (e) $27abc\sqrt{abc}$

7. 1

9. (a) 54 (b) 75 (c) 0.5

11. $v = \frac{fu}{u-f}$

13. $u = \frac{4s-3at^2}{4t}$

15. −16.2

CHAPTER 2

Exercise 2.1

1. 2
3. (a) 204 (b) 3 (c) –44 (d) 95.625
5. 1 842 387
7. 2

Exercise 2.2

1. $2n$
3. $2(p-6q)$
5. 5
7. $2m(q-3p)$
9. $14-5t$
11. $2m+5$
13. $4(a+b)$
15. $6x+23$

Exercise 2.3

1. Not Identity
3. Identity
5. Identity
7. Not Identity
9. Not Identity

Exercise 2.4

1. $2x^2+5x+3$
3. y^2+6y
5. $49n^2-28n+4$
7. $a^2-2ab+b^2$
9. $18p^2+6pq-4q^2$
11. x^2-y^2
13. $(12x^2-4x-1)$ cm^2
15. n^2-1

Exercise 2.5

1. $5(a+b)$
3. $2(m+n)$
5. $b(x+y)$
7. $3(x+2y)$
9. $5(3p-q)$
11. $2s(r+2)$
13. $5k(5l-7t)$
15. $2p(3q-t+4m)$
17. $4xy(z+4n+2r)$
19. $n\left(\frac{1}{2}m+\frac{1}{3}t-\frac{1}{4}x\right)$ or $\frac{n}{12}(6m+4t-3x)$
21. $0.125u(y-4)$

Exercise 2.6

1. $(x+1)(x+2)$
3. $(2x-1)(x-8)$
5. $(t+2)(t+4)$
7. $(d-2)(3d+4)$
9. $(m+1)(m+10)$
11. $(2a-3)(a-1)$
13. $(x+3)(x+7)$
15. $(2x+3)(x-5)$
17. $(12x+39)(x-1)$
19. $(3y-5)(y-2)$
21. $(2x-1)(x+1)$

Exercise 2.7

1. $(x+2)(x+4)$
3. $(2y+7)(y+1)$
5. $(3m+7)(4m+3)$
7. $\left(c+\frac{4}{9}\right)^2$
9. $(y-2)^2$
11. $(w-8)^2$
13. $(3x-1)^2$
15. $(3x+11)(3x-11)$
17. $(2x+y)(2x-y)$
19. $(1+ab)(1-ab)$
21. $(a+b)(a+b-3c)$
23. $k=9$
25. 32 000

Revision exercise 2

1. $-8a-14c$
3. (a) $a^2-2ab-b^2$ (b) $y+2x-7$
5. (a) $a^2+2a-24$ (b) $6x^2-15x+9$
   (c) $2xp+xq+4yp+2yq$ (d) $6a^2+13ab+6b^2$
   (e) $3r^2-15rs+12s^2$
7. Constant term is 6
9. $(2x^2+3x-9)$ square units
11. $\frac{y^2}{16}$ square units
13. (a) Not identity (b) Identity (c) Not identity
    (d) Not identity (e) Not identity (f) Identity
    (g) Identity
15. (a) $(3a-5b)(3a+5b)$ (b) $3(c+1)(c+3)$ (c) $(17y-4x)(16x+7y)$
17. (a) 95 020 (b) 344 000
19. (c) Perfect square (d) Perfect square

CHAPTER 3

Exercise 3.1

1. $x = 0$ or $x = 5$
3. $h = 0$ or $h = 3$
5. $f = 0$ or $f = \frac{11}{4}$
7. $x = 0$ or $x = \frac{7}{5}$
9. $t = 0$ or $t = 4$
11. $x = 0$ or $x = \frac{7}{3}$
13. $x = 0$ or $x = \frac{3}{5}$
15. $x = 0$ or $x = 7$
17. $p = 0$ or $p = 1$
19. $x = 0$ or $x = 4$

Exercise 3.2

1. $x = -1$ or $x = -6$
3. $x = -\frac{3}{2}$ or $x = -3$
5. $x = -3$ or $x = -7$
7. $x = -\frac{1}{2}$ or $x = -\frac{5}{2}$
9. $x = \frac{3}{2}$ or $x = -4$
11. $x = -1$

Exercise 3.3

1. $x = 1$ or $x = 9$
3. $x = 4$ or $x = \frac{-2}{3}$
5. $x = 2$ or $x = \frac{3}{2}$
7. $x = 3$ or $x = 7$
9. $x = -\frac{1}{5}$ or $x = \frac{1}{2}$
11. $x = 2$ or $x = 3$
13. $q = 3$ or $q = -2$
15. $n = 15$ or $n = -8$
17. $q = 7$ or $q = -1$
19. $x = -2$ or $x = 3$

Exercise 3.4

1. $x = \frac{7}{2}$ or $x = -\frac{7}{2}$
3. $y = 2$ or $y = -2$
5. $a = \frac{3}{5}$ or $a = -\frac{3}{5}$
7. $b = \frac{7}{4}$ or $b = -\frac{7}{4}$
9. $x = 1$ or $x = -1$
11. $x = 8$ or $x = -2$
13. $x = 0$ or $x = 10$
15. $c = -\frac{1}{7}$
17. $x = \frac{2}{3}$
19. $r = \frac{1}{2}$

Exercise 3.5

1. 36 will be added; $(x-6)^2$
3. $\frac{49}{16}$ will be added; $\left(x+\frac{7}{4}\right)^2$
5. $\frac{1}{4}$ will be added; $\left(x-\frac{1}{2}\right)^2$
7. 36 will be added; $(p+6)^2$
9. $\frac{1}{16}$ will be added; $\left(t+\frac{1}{4}\right)^2$

Exercise 3.6

1. $x = -5$ or $x = 3$
3. $v = \frac{2}{3}$ or $v = -3$
5. $x = -1 \pm \sqrt{3}$
7. $x = 2 \pm \sqrt{2}$
9. $x = \frac{-5 \pm \sqrt{7}}{3}$
11. $x = \frac{-5 \pm \sqrt{13}}{2}$
13. $x = \frac{-5 \pm \sqrt{17}}{2}$
15. $m = \frac{-11 \pm \sqrt{41}}{2}$
17. $x = \frac{7 \pm \sqrt{5}}{2}$
19. $x = \frac{11 \pm \sqrt{133}}{2}$

Exercise 3.7

1. $x = 1$ or $x = 3$
3. $x = \frac{3 \pm \sqrt{15}}{3}$
5. $x = 3$ or $x = \frac{1}{2}$
7. $x = \frac{1 \pm \sqrt{7}}{3}$
9. $x = \frac{3 \pm \sqrt{57}}{6}$
11. $x = \frac{3}{2}$ or $x = -1$
13. $x = 1$ or $x = -3$
15. $t = 10(1 \pm \sqrt{5})$
17. $x = 0$ or $x = -2$

Exercise 3.8

1. Base = 6 cm
3. 9 and 14
5. Numbers are 11 and 12
7. 6 years
9. 8 cm; 32 cm

Revision exercise 3

1. (a) $x = 0$ or $x = -3$
   (b) $x = 0$ or $x = 5$
   (c) $x = 0$ or $x = \frac{1}{3}$
   (d) $x = 0$ or $x = \frac{3}{2}$
   (e) $x = 0$ or $x = 5$
   (f) $x = 0$ or $x = \frac{3}{7}$
   (g) $x = 5$ or $x = -8$
   (h) $x = 3$ or $x = -\frac{2}{3}$
   (i) $x = -\frac{3}{4}$ or $x = -\frac{1}{3}$
   (j) $x = -1$ or $x = -2$
   (k) $x = 6$ or $x = 4$
   (l) $x = -\frac{3}{2}$ or $x = 2$
   (m) $x = -\frac{1}{3}$ or $x = \frac{2}{3}$
   (n) $x = 2$ or $x = \frac{5}{3}$
   (o) $x = -\frac{5}{2}$ or $x = \frac{5}{2}$
   (p) $y = 6$ or $y = -6$
   (q) $x = 14$ or $x = 2$
   (r) $x = \frac{5}{2}$
   (s) $y = \frac{1}{3}$
   (t) $x = 4$ or $x = -10$

3. (a) $x = \frac{1}{2}$
   (b) $x = \frac{-6 \pm \sqrt{21}}{5}$
   (c) $x = \frac{9 \pm \sqrt{41}}{2}$
   (d) $x = -\frac{2}{3}$ or $x = 2$
   (e) $x = \frac{1}{2}$ or $x = 2$
   (f) $x = 2$ or $x = -\frac{9}{5}$

5. sh 60, sh 100
7. 1.5 cm
9. 11

CHAPTER 4

Exercise 4.1

1. (a) 3.1065×10^4 (b) 9.51×10^1 (c) 9.999×10^3
   (d) 6×10^0 (e) 1×10^{-2} (f) 6.903×10^1
   (g) 2.53009115×10^8 (h) 5.41×10^0 (i) 4.068×10^{-4}
   (j) 7.245×10^0 (k) 1.985×10^3 (l) 8×10^{-6}
   (m) 8×10^{-6} (n) 3.0×10^1 (o) 4.6318×10^2
   (p) 2.65×10^3

3. (a) 9.0×10^9 (b) 2.2×10^7 (c) 2.04×10^0
   (d) 2.0×10^{10} (e) 1.221×10^{-2}

5. $1.2568 \times 10^3\ \text{cm}^2$

7. (a) 0.0000000247 (b) 0.000000339

Exercise 4.2

1. (a) $4 = \log_2 16$ (b) $2 = \log_5 25$ (c) $5 = \log_3 243$

3. (a) $11^2 = 121$
   (b) $10^4 = 10\ 000$ (c) $10^{-1} = 0.1$
   (d) $4^{\frac{1}{2}} = 2$ (e) $2^{-2} = 0.25$ (f) $\left(\frac{1}{5}\right)^{-3} = 125$

5. $\frac{1}{125}$

Exercise 4.3

1. (a) 6 (b) 7 (c) 6 (d) –1

3. (a) 3 (b) 2 (c) 4
   (d) 3 (e) 1 (f) $\frac{2}{3}$

5. (a) $x = 20$ (b) $x = 10$

7. (a) $x = 45$ (b) $x = 15$

9. $y = \frac{1000}{x^2}$

Exercise 4.4

1. (a) 0.1995 (b) 0.4742 (c) 0.6116 (d) 0.9316
   (e) 0.9545 (f) 0.7782 (g) 0.5371 (h) 0.8771

3. (a) 3.9407 (b) 2.8452 (c) 4.8832 (d) 6.6335
   (e) 1.9227 (f) 5.0464 (g) 1.3010 (h) 3.2980
   (i) 4.8926 (j) 2.5490 (k) 1.6401 (l) 5.6928

5. (a) Characteristics 3, mantissa 6 156
   (b) Characteristics 1, mantissa 8 937
   (c) Characteristics 0, mantissa 4 514
   (d) Characteristics 8, mantissa 0 000

Exercise 4.5

1. (a) $\overline{1}.2807$ (b) $\overline{2}.3817$ (c) 0.3817
   (d) 1.0151 (e) $\overline{3}.3402$ (f) 0.3817

Exercise 4.6

1. (a) $\overline{1}.8338$ (b) $\overline{3}.9031$ (c) $\overline{1}.8692$ (d) $\overline{5}.6522$
   (e) $\overline{2}.4942$ (f) $\overline{2}.1697$ (g) $\overline{1}.0969$ (h) $\overline{3}.9917$
3. (a) Characteristics –2, mantissa 3 201
   (b) Characteristics –1, mantissa 7 911
   (c) Characteristics –6, mantissa 8 762
   (d) Characteristics –2, mantissa 8 505
5. (a) 3.1882 (b) $\overline{3}.1882$ (c) $\overline{3}.3542$
7. (a) $\overline{2}.9722$ (b) $\overline{1}.8136$ (c) $\overline{2}.8864$ (d) 0.7335
9. 0.6386
11. (a) $x = 848.1$ (b) $x = 50$

Exercise 4.7

1. 21 910   3. 0.2381   5. 0.008433

7. 87.54   9. 60 530   11. 0.0000000208

13. 31.01   15. 0.00006098   17. 375 00000

19. 3.034

Exercise 4.8

1. 323   3. 5.12   5. 0.4529

7. 11 530 000 000   9. 2.002   11. $R = 0.121$

13. d = 183.7   15. 14.55   17. 7.234

19. 284.6 cm^3

21. (a) $\bar{2}.6149$ (b) 0.8457   (c) 1.1553   (d) $\bar{4}.6522$

Revision exercise 4

1. (a) 8.419×10^6 (b) 4.57×10^1 (c) 7.16×10^2
   (d) 1.23×10^{-4} (e) 4×10^0 (f) 5×10^{-3}

3. (a) 2.2×10^{12} (b) 1.0×10^{-4} (c) 1.6×10^2
   (d) 1.44×10^0

5. (a) $x = 4\,096$ (b) $x = 5$ (c) $x = 1\,000$

7. (a) $x = 10^6$ or 1 000 000   (b) $x = 6^6$ or 46 656

9. 1.24304

11. (a) 1   (b) 2

13. (a) 20970   (b) 358.3

15. (a) 108.2   (b) 2.1842

17. (a) 0.00000246   (b) 0.001784

19. T = 1.8223

21. S = 125.49

23. 183.9

25. 3 or 27

CHAPTER 5

Exercise 5.1

1. Use degree measure of a straight line:

$A\hat{B}D + A\hat{B}C = 180^\circ$ and $A\hat{C}B + A\hat{C}E = 180^\circ$

Thus, $A\hat{B}D + A\hat{B}C = A\hat{C}B + A\hat{C}E$

But, $A\hat{B}C = A\hat{C}B = a$

$\therefore A\hat{B}D = A\hat{C}E$

3. Use degree measure of a straight line:

$x + a = 180^\circ$, $c + y = 180^\circ$

$x + a = c + y$

But $a = c$

$\therefore x = y$

5. Use alternate interior angles:

$E\hat{D}C = X\hat{C}D$, $X\hat{C}B = A\hat{B}C$

$E\hat{D}C = X\hat{C}D$

$X\hat{C}B = C\hat{B}A$

$\therefore B\hat{C}D = E\hat{D}C + C\hat{B}A$

7. $P\hat{Q}S + R\hat{Q}S = 180^\circ$ (straight angle)

Also, $Q\hat{R}S + R\hat{S}Q + S\hat{Q}R = 180^\circ$

Sum of interior angles of any triangle is 180°

Hence, $P\hat{Q}S + R\hat{Q}S = Q\hat{R}S + R\hat{S}Q + S\hat{Q}R$

$= 180^\circ$

So that, $P\hat{Q}S = Q\hat{R}S + R\hat{S}Q$

$2a = Q\hat{R}S + a$

$Q\hat{R}S = a$

but $Q\hat{R}S = b$

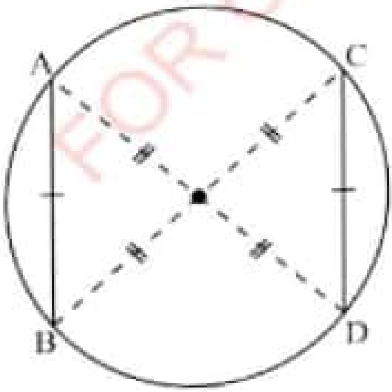
$\therefore a = b$

9. Given triangles DBC, DBE, ABE and PEC
then,
$E\hat{D}B + D\hat{B}E + B\hat{E}D = 180°$ ...... (i) (sum of interior angles of triangle DBE)
$C\hat{A}B + A\hat{B}C + B\hat{C}A = 180°$ ......(ii) (sum of interior angles of triangle ABC)
Now,
$x + D\hat{B}E + b = 180°$ ....(iii) (from equation (i))
$a + A\hat{B}C + y = 180°$ ....(iv) (from equation (ii))
Then, equating equation (iii) and (iv), we obtain
So that, $x + b = a + y$
but $x = y$
$\therefore a = b$

Exercise 5.2

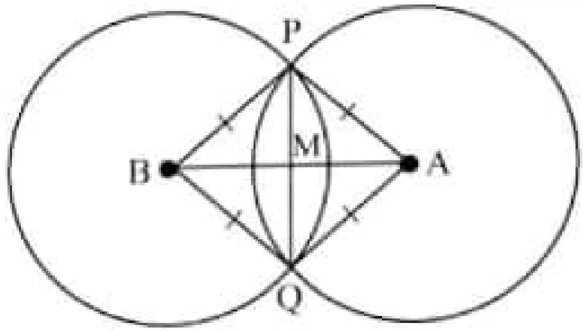
1. $\overline{AC}$ is common
$\left.\begin{array}{l}\overline{AD} = \overline{AB} \\ \overline{CD} = \overline{BC}\end{array}\right\}$ (given)
So, $\Delta A\hat{B}C = \Delta A\hat{D}C$ (by SSS)

3. $\left.\begin{array}{l}\overline{AF} = \overline{ED} \\ \overline{EB} = \overline{CF} \\ \overline{AB} = \overline{CD}\end{array}\right\}$ (given)
$\therefore B\hat{A}F = C\hat{D}E$ (by SSS)

5.


$\overline{AB} = \overline{CD}$ (given)
$\overline{AO} = \overline{OD} = \overline{OC} = \overline{OB}$ (radii)
$\overline{AD} = \overline{BC}$ (diameter)
$\therefore A\hat{O}B = C\hat{O}D$ (by SSS)

7.



$\overline{AP} = \overline{AQ} = \overline{PB} = \overline{BQ}$ (radii)

$\overline{PM} = \overline{MQ}$ (given)

$\overline{AM} = \overline{MB}$ (given)

$\overline{AB}$ is common

Then, $\overline{AB}$ bisects $P\hat{A}Q$ (by SSS)

9. $\left.\begin{matrix}\overline{AD} = \overline{BC} \\ \overline{AC} = \overline{BD}\end{matrix}\right\}$ (given)

$\therefore \Delta A\hat{B}D \cong \Delta C\hat{A}B$ (by SSS)

11. $\overline{AB} = \overline{CD}$ (given)

$\overline{AD} = \overline{CB}$ (given)

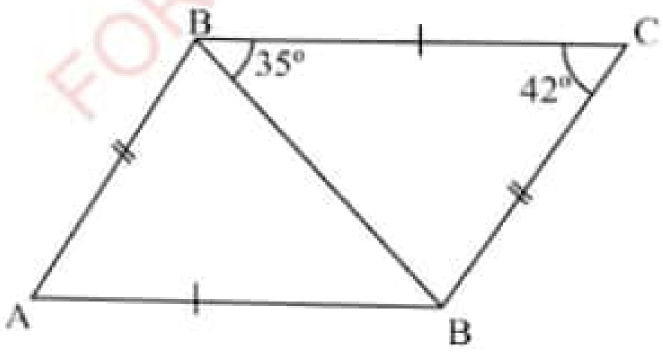
$B\hat{D}C + B\hat{C}D + C\hat{B}D = 180°$

$35° + 42° + C\hat{B}D = 180°$

$C\hat{B}D = 103°$

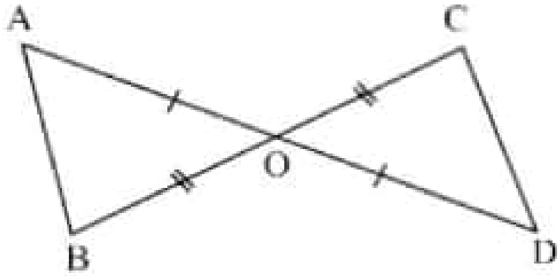
$D\hat{B}A = C\hat{D}B$ because $\overline{CD} // \overline{AB}$

$\therefore A\hat{B}D = 35°$.



Exercise 5.3

1.



(a) $\left.\begin{array}{l}\overline{AO}=\overline{OD}\\ \overline{OB}=\overline{OC}\end{array}\right\}$ (given)

From the figure, $A\hat{B}O = O\hat{C}D$

$\therefore \overline{AB} = \overline{CD}$ (by SAS)

(b) $B\hat{A}O = C\hat{D}O$

3. $\overline{AX} = \overline{DX}$ (given)

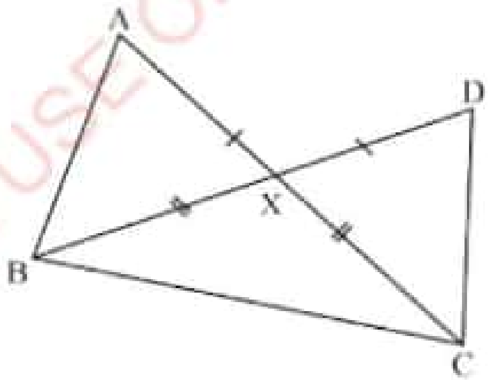
$\overline{BX} = \overline{CX}$ (given)

In ΔABC and ΔDBC

$\overline{BC}$ is common

$B\hat{X}A = C\hat{X}D$ (given)

$\therefore B\hat{A}C = C\hat{D}B$ (by SAS)



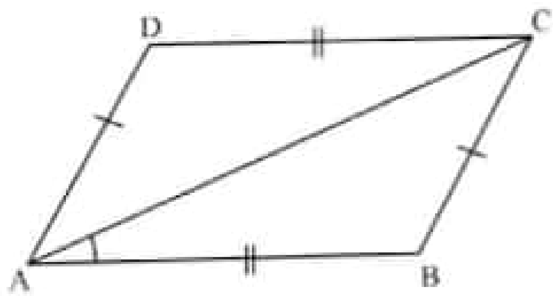
5. In ΔABC and ΔADC

$\overline{AB} = \overline{DC}$ (given)

$\overline{AC}$ is common

$C\hat{A}D = C\hat{A}B$ (given)

$\overline{AD} = \overline{BC}$ (by SAS)



Math F2 - (BOOK) (DUMMY) Sept 4.indb 387

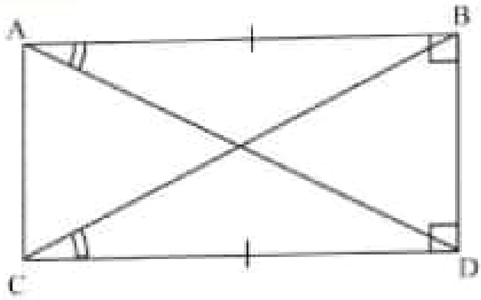
7. $\overline{AB} = \overline{DC}$ (given)

$A\hat{B}D = B\hat{D}C$ (given)

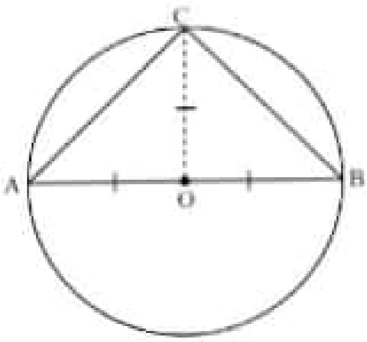
In ΔABD and ΔBCD

$\overline{BD}$ is common

$\therefore D\hat{A}B = B\hat{C}D$ (by SAS)



9.



$\overline{AO} = \overline{OB} = \overline{OC}$ (given)

$A\hat{O}C = B\hat{O}C$ (given)

$\overline{OC}$ is common

$\therefore \overline{AC} = \overline{CB}$ (by SAS)

11. $D\hat{A}C + A\hat{C}D + A\hat{D}C = 180°$

$55° + 70° + A\hat{D}C = 180°$

$\therefore A\hat{D}C = 180° - 125°$

$D\hat{A}C = 55°$.

Exercise 5.4

1. From the figure,

$D\hat{A}C = B\hat{A}C$

$D\hat{C}A = B\hat{C}A$

$\overline{AC}$ is common

$\therefore \overline{AB} = \overline{AD}$ (by AAS)

3. $\overline{AO} = \overline{OD}$ (given)

$A\hat{B}O = B\hat{C}D$ (alternate interior angle as, $\overline{AB} // \overline{CD}$)

$B\hat{A}O = D\hat{C}O$ (alternate interior angle as, $\overline{AB} // \overline{CD}$)

$\therefore \overline{AB} = \overline{CD}$ (by AAS)

5. $\overline{XZ} // \overline{YR}$ (given)

$\overline{QZ} = \overline{ZR}$ (given)

$\overline{PY} = \overline{YR}$ (given)

$Z\hat{X}Y = X\hat{Y}P$ (alternate interior angle as, $\overline{XZ} // \overline{PR}$)

$Q\hat{Z}X = Z\hat{X}Y$ (alternate interior angle as, $\overline{QZ} // \overline{XY}$)

$\therefore \overline{XY} = \overline{QZ}$ (by AAS)

7. From the figure,

$\overline{AC}$ is common

$D\hat{A}C = B\hat{A}C$

$A\hat{B}C = A\hat{D}C$

$\therefore \overline{AB} = \overline{AD}$ (by AAS)

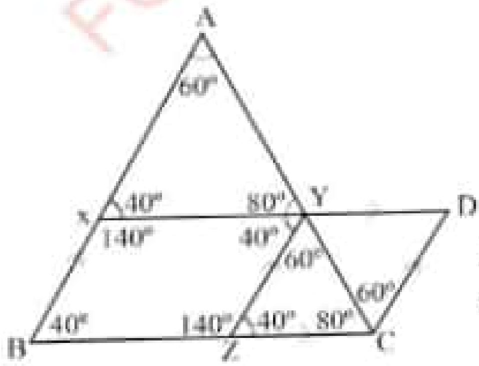
9. $A\hat{B}C = B\hat{A}D$ (given)

$B\hat{C}A = A\hat{D}B$ (given)

$\overline{AB}$ is common

$\therefore \overline{BC} = \overline{AD}$ (by AAS)

11.



$A\hat{X}Y = X\hat{Y}Z$ (alternate interior angle)

$Y\hat{Z}C = 40°$

Exercise 5.5

1. $\overline{AC} = \overline{BD}$ (given)

   $D\hat{A}B = C\hat{B}A = 90^\circ$ (given)

   $\overline{AC}$ and $\overline{BD}$ are common sides,

   $\therefore \Delta ABD \cong \Delta BAC$ (RHS)

3. $\overline{BF} = \overline{CE}$ (given)

   $B\hat{E}C = B\hat{F}C = 90^\circ$ (given)

   $\overline{BC}$ is common

   $\therefore \overline{CF} = \overline{BE}$ (RHS)

5. $\Delta OPA \cong \Delta OPB$ (RHS)

   $\overline{OP}$ is common

   $\overline{OA} = \overline{OB}$ (given)

   $O\hat{A}P = P\hat{B}O = 90^\circ$ (given)

   $\therefore \overline{PA} = \overline{PB}$ (RHS)

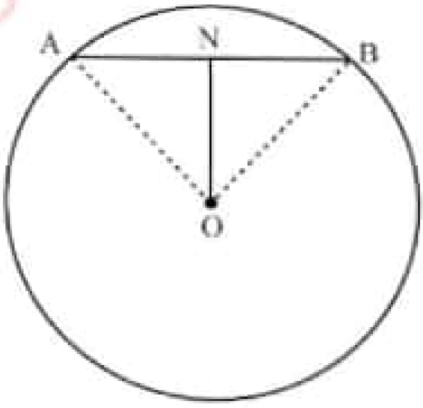
7. $\overline{OB} = \overline{OA}$ (radii of the circle)

   $O\hat{N}A = O\hat{N}B = 90^\circ$ (given)

   $\overline{ON}$ is common

   $\Delta OAN \cong \Delta ONB$ (RHS)

   $\therefore \overline{AN} = \overline{NB}$



9. $\overline{AC} = \overline{BC}$ (given)

   $\overline{DE} = \overline{FE}$ (given)

   $C\hat{D}E = F\hat{E}B = 90^\circ$ (given)

   $A\hat{D}E = C\hat{D}E = 90^\circ$ (given)

   $\Delta ECD \cong \Delta FEB$ (RHS)

   $\therefore \overline{CD} = \overline{EB}$

Revision exercise 5

1. $\overline{BA} = \overline{BC}$ (given)

   $\overline{KA} = \overline{KC}$ (given)

   $\overline{BK}$ is common

   $\therefore B\hat{A}K = B\hat{C}K$ (by SSS)

3. $M\hat{O}L = P\hat{O}L = 90^\circ$ (given)

   $L\hat{M}O = L\hat{P}O$ (given)

   $\overline{OL}$ is common

   $\therefore \overline{ML} = \overline{PL}$ (by AAS)

5. $\overline{AB} = \overline{AC}$ (given)

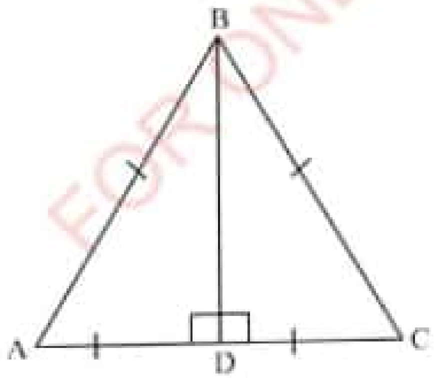
   $\overline{BM} = \overline{MC}$ (given)

   $\overline{AM}$ is common

   $A\hat{M}C = B\hat{M}A$ (by SSS)

   $AM \perp BC$

   $\therefore A\hat{M}C = B\hat{M}A = 90^\circ$

7. 

   $\overline{AB} = \overline{BC}$ (given)

   $\overline{BD}$ is common

   $B\hat{D}A = B\hat{D}C = 90^\circ$ (given)

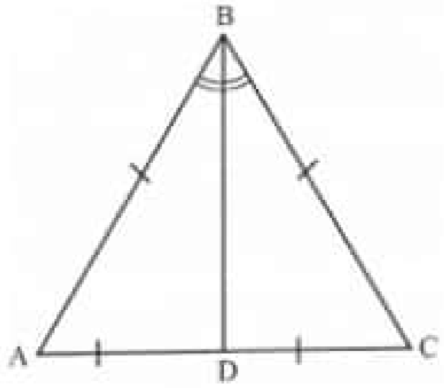
   $\overline{AC}$ is the base of ΔABC

   $\Delta ADB = \Delta BDC$

   $\overline{BD}$ is common (bisector)

   $\therefore \overline{BD}$ is perpendicular to $\overline{AC}$

9.



$\overline{BD}$ is common

$\overline{AB} = \overline{BC}$ (given)

$A\hat{B}D = D\hat{B}C$ (given)

$\overline{AD} = \overline{CD}$ (by SAS)

$\therefore \overline{BD}$ is perpendicular to $\overline{AC}$ at its midpoint.

11. $L\hat{A}P = M\hat{A}P$ (given)

$\overline{LP} = \overline{PM}$ (given)

$\overline{AP}$ is common

$\therefore \overline{AP}$ bisect $L\hat{A}M$ (by SSS)

13. $\overline{DC} = \overline{AD}$ (given)

$\overline{BC} = \overline{AB}$ (given)

$\overline{BD}$ is common

$\therefore B\hat{A}D = B\hat{C}D$ (by SSS)

15. $D\hat{A}X = C\hat{B}X$ (given)

$\overline{AD} // \overline{XC}$ (given)

$\overline{CD} // \overline{AX}$ (given)

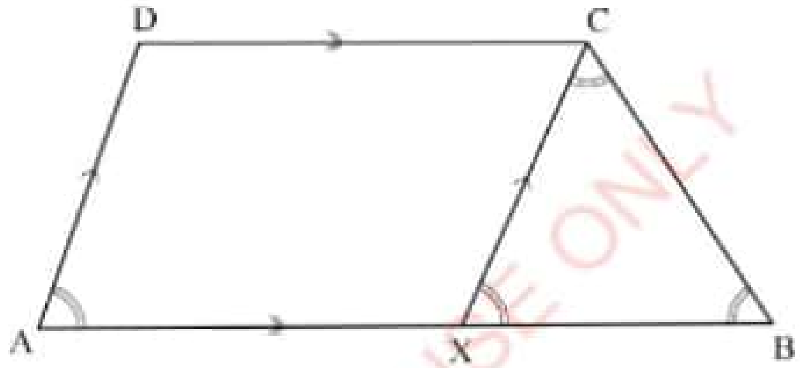
$D\hat{C}X = C\hat{X}B$ (alternate interior angles $\overline{CD} // \overline{AX}$)

ADCX is a quadrilateral such that

$D\hat{A}X = X\hat{C}D$ and $A\hat{D}X = A\hat{X}C$

$C\hat{X}B = X\hat{B}C = D\hat{A}X$

$\therefore B\hat{X}C$ is an isosceles triangle (by AAS)



17. $Q\hat{R}P = Q\hat{P}R = P\hat{Q}R = 60°$

$R\hat{Q}S = 120°$

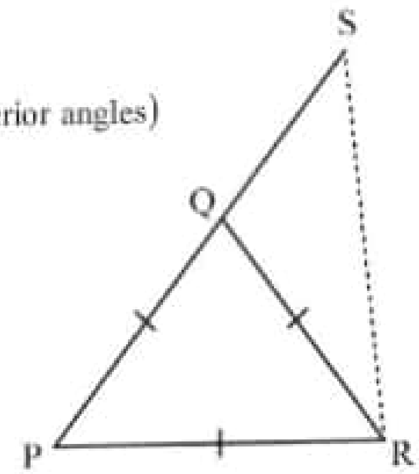
ΔQRS is an isosceles triangle

$120° + Q\hat{R}S + Q\hat{S}R = 180°$ (sum of interior angles)

$Q\hat{R}S + Q\hat{S}R = 60°$

But, $Q\hat{R}S = Q\hat{S}R$

$\therefore Q\hat{R}S = 30°$



Math F2 – (BOOK) (DUMMY) Sept 4.indb 393

CHAPTER 6

Exercise 6.1

1. (a) $P\hat{Q}R = T\hat{S}M$, $Q\hat{R}P = S\hat{M}T$, $Q\hat{P}R = S\hat{T}M$

$\overline{PQ}$ corresponds to $\overline{TS}$; $\overline{SM}$ corresponds to $\overline{QR}$.
and $\overline{PR}$ corresponds to $\overline{TM}$

(b) $L\hat{M}N = A\hat{B}C$, $M\hat{N}L = B\hat{C}A$ and $N\hat{L}M = C\hat{A}B$

$\overline{LM}$ corresponds to $\overline{AB}$, $\overline{MN}$ corresponds to $\overline{BC}$ and $\overline{NL}$ corresponds to $\overline{CA}$.

3. (a) $A\hat{C}B = 40^\circ$ (b) $A\hat{C}B = 50^\circ$ (c) $A\hat{C}B = 90^\circ$

5. (a) $\Delta DEF \sim \Delta ABC$, proportion is $\frac{1}{2}$ (by SSS)

(b) $\Delta BCD \sim \Delta CAD$, proportion is $\frac{4}{3}$

(c) $\Delta ABD \sim \Delta CDE$, proportion is $\frac{1}{2}$ (by SSS)

7. Yes, the information is sufficient.

$C\hat{A}T = B\hat{D}T$ alternate angles.

$A\hat{C}T = D\hat{B}T$ alternate angles

$A\hat{T}C = D\hat{T}B$ vertically opposite angles

Therefore, $\Delta ACT \sim \Delta DBT$ (corresponding angles are equal)

9. $\Delta XYZ \sim \Delta LNM$

Exercise 6.2

1. (a) $\Delta BEA \sim \Delta BCD$ (SAS – Similarity Theorem)

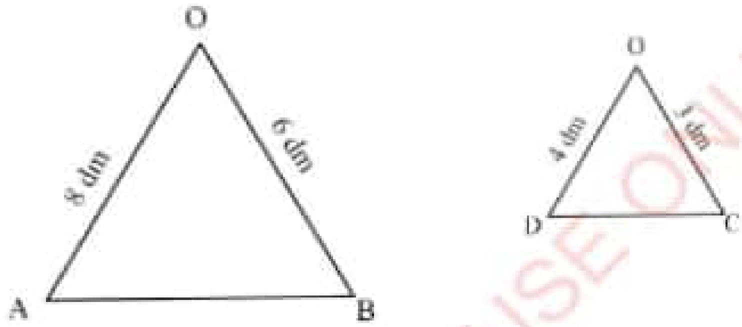
   (b) $\Delta PQR \sim \Delta BAC$ (SSS – Similarity Theorem)

   (c) $\Delta DEF \sim \Delta ZXY$ (AA – Similarity Theorem)

   (d) $\Delta LMN \sim \Delta UTS$ (SAS – Similarity Theorem)

   (e) $\Delta ABC \sim \Delta DEF$ (AA – Similarity Theorem)

3.



Proportionality ratio is $\frac{\overline{OD}}{\overline{OA}} = \frac{\overline{OC}}{\overline{OB}} = \frac{1}{2}$

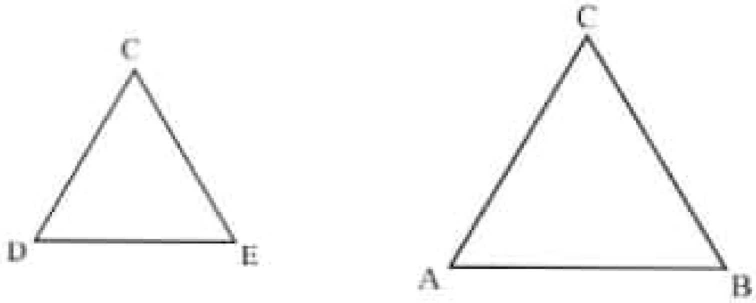
$A\hat{O}B$ corresponds to $D\hat{O}C$

$O\hat{A}B$ corresponds to $O\hat{D}C$

$A\hat{B}O$ corresponds to $D\hat{C}O$

$\therefore \Delta ABO \sim \Delta OCD$

5.



$\Delta CDE \sim \Delta CAB$ (given)

$D\hat{C}E$ corresponds to $A\hat{C}B$

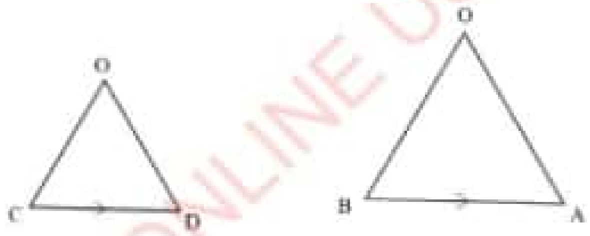
$D\hat{E}C$ corresponds to $A\hat{B}C$

$C\hat{D}E$ corresponds to $B\hat{A}C$

$\therefore \overline{DE} // \overline{AB}$

7. $\overline{CD} // \overline{AB}$ given

(a) $C\hat{O}D = A\hat{O}B$ (vertically opposite angles)



$C\hat{O}D$ corresponds to $B\hat{O}A$

$C\hat{D}O$ corresponds to $B\hat{A}O$

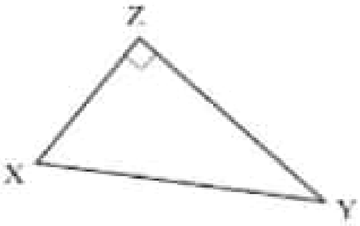
$D\hat{C}O$ corresponds to $A\hat{B}O$

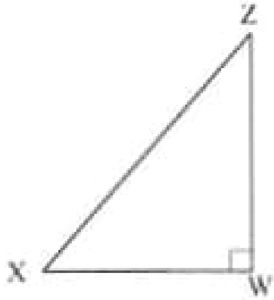
$\Delta ABO \sim \Delta DCO$

(b) Since $\Delta ABO \sim \Delta DCO$, then the proportionality ratio is given by

$$\frac{\overline{AO}}{\overline{OD}} = \frac{\overline{BO}}{\overline{OC}}$$

9.





$X\hat{Z}Y = X\hat{W}Z = 90^\circ$ (right angled triangle)

$\frac{\overline{XW}}{\overline{XZ}} = \frac{\overline{XZ}}{\overline{XY}}$ (given)

$X\hat{Z}Y$ corresponds to $X\hat{W}Z$

$X\hat{Y}Z$ corresponds to $X\hat{Z}W$

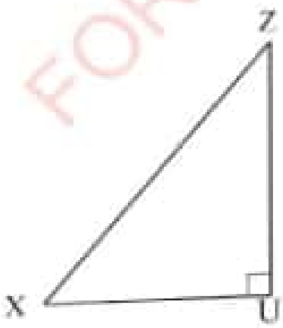
$Y\hat{X}Z$ corresponds to $W\hat{X}Z$

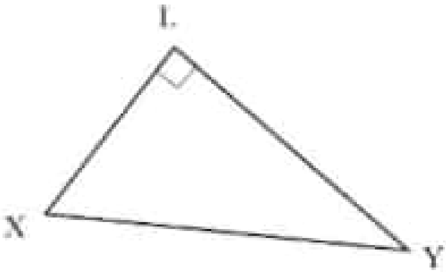
$\Delta XWZ \sim \Delta ZWY$

$\overline{ZW}$ is common

$\therefore \overline{ZW}$ is perpendicular $\overline{XY}$.

11. $\Delta LXY \sim \Delta UZX$





Exercise 6.3

1. (a) $\overline{BC} = 20$ dm (b) $\overline{ED} = 8$ dm (c) $\overline{BD} = 7$ dm
   (d) $\overline{AD} = 12$ dm (e) $\overline{AB} = 35$ cm (f) $\overline{CE} = 4$ cm

3. (a) $\Delta XYZ \sim \Delta CBA$
   $\overline{YZ} = 20$ cm, $\overline{BC} = 6$ cm, $A\hat{B}C = X\hat{Y}Z = 90^\circ$ and $\overline{AC} = 15$ cm
   (b) $\Delta XYZ \sim \Delta MLN$
   $\overline{XY} = 2$ cm, $\overline{MN} = 30$ cm, $X\hat{Z}Y = 49^\circ$
   (c) $\Delta XYZ \sim \Delta FED$
   $\overline{XY} = 5$ cm, $\overline{EF} = 10$ cm, $\overline{DE} = 8$ cm
   $X\hat{Y}Z = 53^\circ$

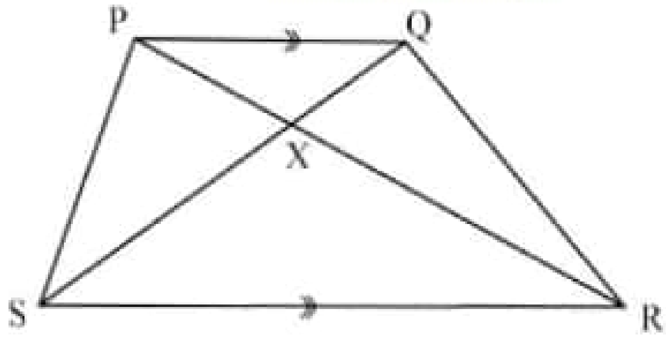
5. (a) $\Delta YOQ \sim \Delta XOP$
   (b) $\overline{OY} = 5.25$ m, $\overline{OQ} = 3$ m

Revision exercise 6

1. (a) $A\hat{E}B = D\hat{E}C$ (common)
   $E\hat{B}A = E\hat{C}D$ (both right angles)
   The proportional sides needed to show that they are similar are
   $\frac{\overline{AE}}{\overline{DE}}, \frac{\overline{EB}}{\overline{EC}}$ or $\frac{\overline{BA}}{\overline{CD}}$.

3. $L\hat{M}N = A\hat{B}C$, $M\hat{N}L = B\hat{C}A$, $N\hat{L}M = C\hat{A}B$
   $\overline{PQ}$ is proportional to $\overline{AB}$, $\overline{QR}$ is proportional to $\overline{BC}$ and $\overline{PR}$ is proportional to $\overline{CA}$, proportional ratio is 1:1.

5.



$\overline{PQ} // \overline{RS}$ (given)

$\overline{PQ}$ is common in ΔPQS and ΔPQR

$S\hat{X}R = P\hat{X}Q$ (vertically opposite angles)

$\therefore \frac{\overline{PX}}{\overline{XR}} = \frac{\overline{QX}}{\overline{XS}}$

7. $\overline{DE} = 5$ dm

9. ΔABC is equilateral, ΔMNP is equilateral

$\therefore \overline{AB} = \overline{BC} = \overline{AC}$ and $\overline{MN} = \overline{MP} = \overline{NP}$

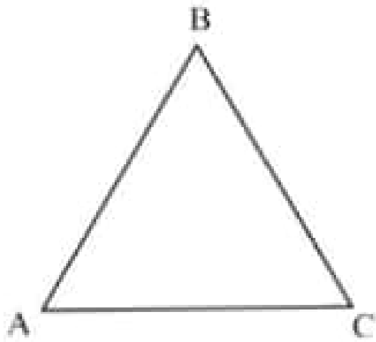
$A\hat{B}C$ corresponds to $M\hat{N}P$

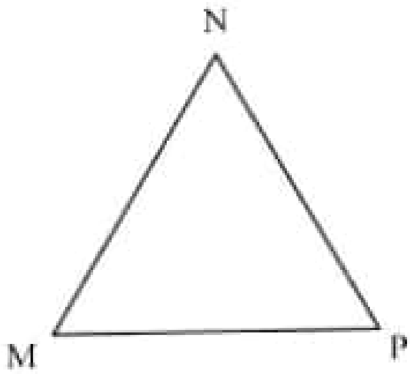
$A\hat{C}B$ corresponds to $M\hat{N}P$

$B\hat{A}C$ corresponds to $M\hat{N}P$

Then, $\frac{\overline{AB}}{\overline{MN}} = \frac{\overline{AC}}{\overline{MP}} = \frac{\overline{BC}}{\overline{NP}}$

$\therefore \Delta ABC \sim \Delta MNP$



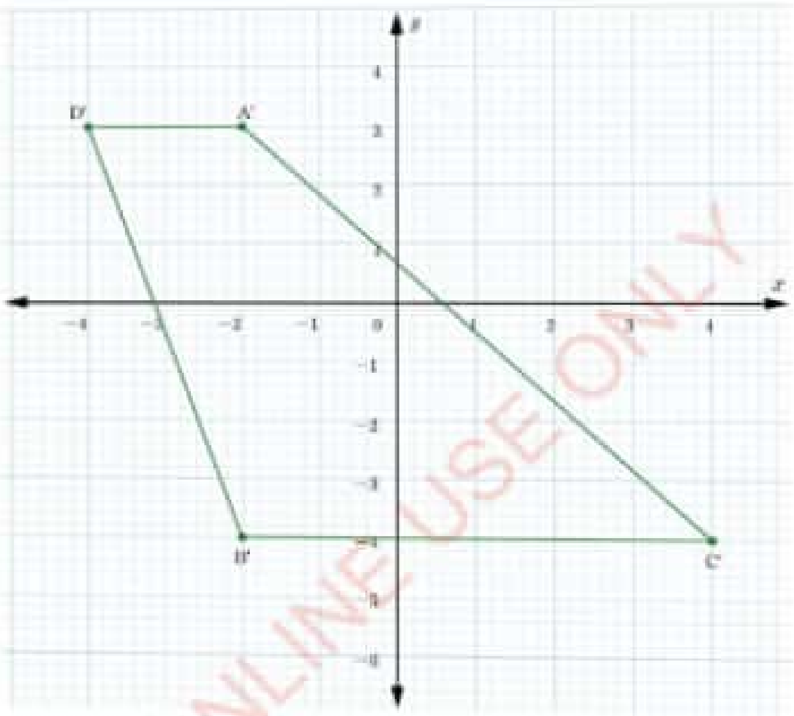


CHAPTER 7

Exercise 7.1

1. D′(4, –2)
3. Q′(4, 3)
5. (–2, –1)
7. (–1, –2)
9. A′(5, –2)
11. R(2, 9)
13. (a) A′(–2, 3), B′(–2, –4), C′(4, –4) and D′(–4, 3)

 (b)



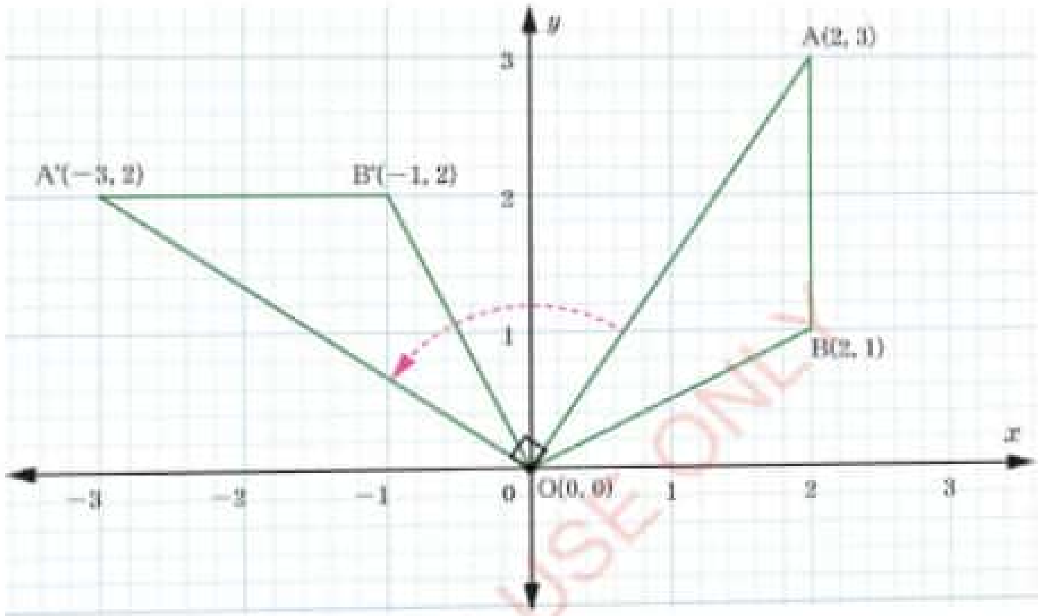
15. Reflection in the x – axis

Exercise 7.2

1. (–1, –2) 3. (–1, –2) 5. (2, –1)

7. (5, 0)

9.



Therefore, the co – ordinates of its image are O′(0,0), A′(–3, 2) and B′(–1, 2).

Exercise 7.3

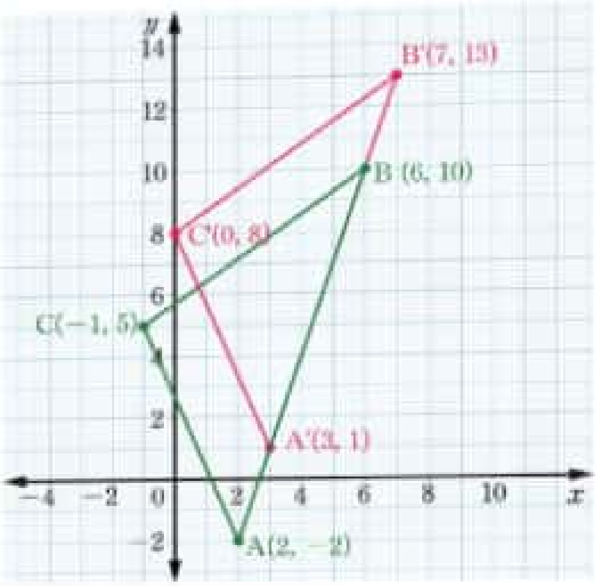
1. (a) (–8, 11) (b) (3, 9)

3. (a) (–7, –7) (b) (–2, –2)

5. (a) C (b) D

7. $(x + h,\ y + k)$

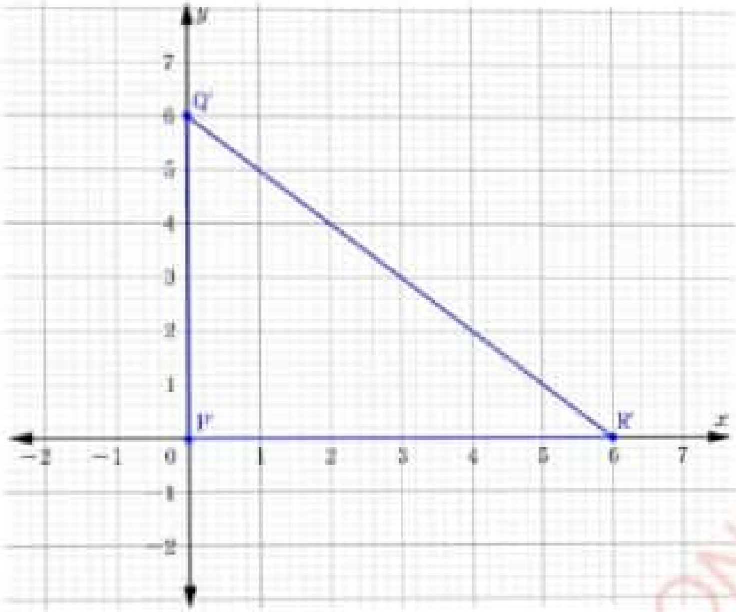
9.



11.

Exercise 7.4

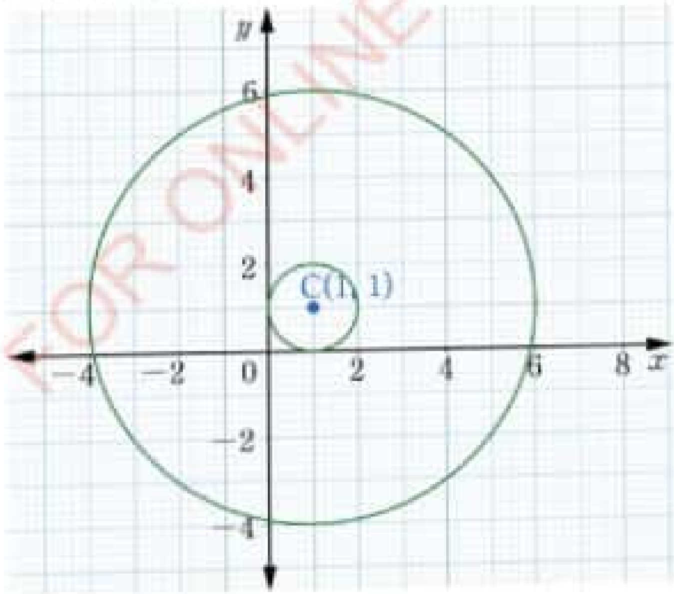
1. Vertices for the image are P′(0, 0), Q′(0, 6) and R′(6, 0).



3. $7\frac{1}{2}$ cm

5. (24, 8)

7. Concentric circle with centre (1, 1) and radii 5 units respectively.



9. (a) $\overline{AD}$ (b) $\overline{AD}$

Exercise 7.5

1. (a) ΔADE (b) $\frac{8}{5}$

3. No, since the widths are not necessarily in the same proportion as the lengths.

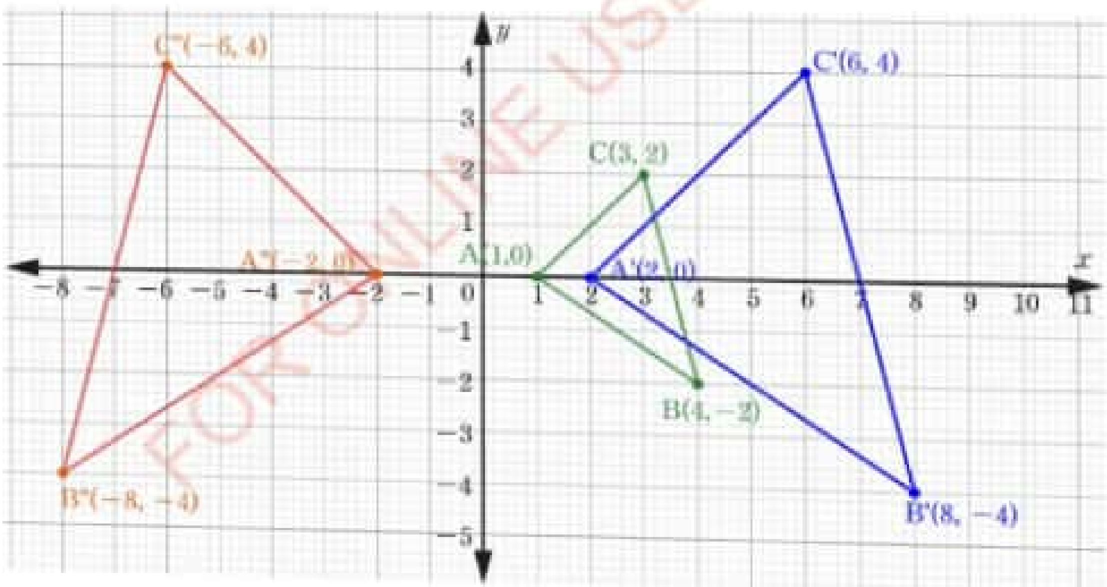
5. $x = 10$; Scale of enlargement 2.

7. $\overline{AB} = 16$ cm

Exercise 7.6

1. (a) A″(2, –4) (b) B″(–1, 0) (c) C″(–4, –4)

3. Given ΔABC with vertices
A(1, 0), B(4, –2) and C(3, 2) vertices of images are
A′ (2, 0), B′(8, –4), C′(6, 4), and
A″ (–2, 0), B″(–8, –4), C″(–6, 4).



Revision exercise 7

1. Reflection, Rotation and Translation
3. B′(–3, –6)
5. Q′(8, 6)
7. (–6, –2)
9. (2, 1)
11. (a) $P\hat{Q}R = 80^\circ$ (b) $Q\hat{R}P + R\hat{Q}P = 140^\circ$
13. (a) Actual length is 40 m or 4 000 cm.
    (b) Actual width is 32 m or 3 200 cm
    (c) Area of the field is 1 280 square metres
15. 4 metres

CHAPTER 8

Exercise 8.1

1. (a) $x = 15$ cm (b) $p = 20$ cm
   (c) $q = 37$ cm (d) $r = 9.8$ cm
3. 9.9 cm
5. $\overline{AC} = 65$ cm, $\overline{AD} = 56$ cm and $\overline{EC} = 63$ cm
7. 8.6 cm
9. 5.8 cm
11. length is 12 cm, Area of ΔABC is 60 cm^2
13. Draw quadrilateral ABCD, where $B\hat{A}D = 90^\circ$

$$\left.\begin{array}{l}(\overline{AB})^2 + (\overline{AD})^2 = (\overline{BD})^2 \\ (\overline{CB})^2 + (\overline{CD})^2 = (\overline{BD})^2\end{array}\right\} \text{given}$$

Then, this shows that, $B\hat{C}D = 90^\circ$

15. $x = 20.8$ cm
17. 23.6 m
19. a, b, d and f
21. 12.99 cm

Exercise 8.2

1. (a) $x = 14.4$ cm (b) $x = 13$ m
3. 400 m
5. Yes, 5.7cm

Revision exercise 8

1. (a) $a = 6.71$ cm (b) $b = 5.66$ cm
   (c) $a = 4$ cm (d) $a = 4.90$ cm $a = 4.90$ cm.
   (e) $a = 7.94$ cm (f) $b = 12.69$ cm
3. 18.60 cm
5. 25.46 cm
7. (a) $a = 7.9$ cm (b) $h = 7.3$ cm
9. 15 m

CHAPTER 9

Exercise 9.1

1. (a) $\tan \hat{M} = \frac{4}{3}$ (b) $\sin \hat{M} = \frac{4}{5}$ (c) $\cos \hat{M} = \frac{3}{5}$

3. (a) $\overline{RT} = 5$ cm (b) $\cos \hat{R} = \frac{4}{5}$ (c) $\sin \hat{R} = \frac{3}{5}$

5. (a) $\tan x = 2.5$ (b) $\sin y = 0.6$

7. (a) $\sin x = \frac{8}{17}$ (b) $\tan x = \frac{8}{15}$

9. (a) $\sin \hat{C} = \frac{c}{b}$ (b) $\cos \hat{C} = \frac{a}{b}$ (c) $\tan \hat{C} = \frac{c}{a}$

   (d) $\frac{\sin \hat{C}}{\cos \hat{C}} = \frac{c}{a}$ (e) $\tan \hat{C} = \frac{\sin \hat{C}}{\cos \hat{C}}$

Exercise 9.2

1. (a) 3 (b) 2 (c) 3 (d) $-\frac{3}{2}$
3. 8 m
5. (a) $\sqrt{3}$ (b) $\frac{\sqrt{3}}{3}$

Exercise 9.3

1. (a) 0.8290 (b) 0.7265 (c) 0.9994
   (d) 0.4344 (e) 0.4412 (f) 2.7475

3. (a) $x =$ 9.192 cm; $y =$ 7.7136 cm
   (b) $x =$ 1.7335 m; $y =$ 2.5700 m
   (c) $x =$ 0.5 m; $y =$ 0.866 m
   (d) $x =$ 8.29 cm; $y =$ 5.592 cm
   (e) $x =$ 1.299 cm; $y =$ 4.206 cm

5. N16° 42′E 7. 47° 12′ 9. 40.5 cm

Exercise 9.4

1. 33° 3. 13.49 m 5. 22.6 m 7. 10.14 m

Revision exercise 9

1. (a) 0.8660 (b) 0.8785 (c) 2.0323
3. $x = 3$ cm, $a = 36°\ 52'$
5. $6\sqrt{3}$ m
7. 534.8 m
9. 46.985 m
11. (a) $h = 12.61$
    (b) $p = 10.971$, $m = 33°\ 12'$
    (c) $n = 56°\ 42'$ $a = 10.59$
    (d) $r = 14.28$, $x = 49°\ 58'$, $m = 17.43$, $x = 50°$
13. (a) 10.6 m (b) 56° 26′

CHAPTER 10

Exercise 10.1

1. A = {1, 3, 5, 7, 9}
3. C = (2, 3, 5, 7, 11}
5. B = {is a set of odd numbers less than twelve}
7. C = {is a set of prime numbers less than twenty}
9. A = {x : x = 2n; where n∈N}
11. A = {x : x = n; where n = 1, 2, 3, ..., 19}
13. (a) It is a set because all members are numbers.
    (b) It is not a set because members have no common characteristics.
    (c) It is a set because all members are vowels.
    (d) It is a set because all members are continent.
    (e) It is not a set because members have no common characteristics.

Exercise 10.2

1. (a) Finite (b) Finite (c) Infinite (d) Empty
   (e) Infinite (f) Empty (g) Infinite (h) Finite
   (i) Empty
3. (a) A = B
   (b) D = F

Exercise 10.3

1. (a) { }, {1} (b) Ø
   (c) { }, {Tito}, {Juma} and {Tito, Juma}
3. B ⊂ A
5. C ⊂ A
7. (a) F (b) F (c) T (d) T
9. C, D and F
11. (a) U = {all english alphabets} (b) U = {all integers}
13. (a) B ⊂ A (b) D ⊂ C

Exercise 10.4

1. $A \cup B = \{5, 10, 15, 20\}$
   $A \cap B = \{15\}$

3. $A \cup B = \{a, b, c, d, e\}$
   $A \cap B = \{a, b, c, d, e\}$

5. $A \cup B = \{\text{cup, spoon}\}$
   $A \cap B = \{\text{cup}\}$

7. $A \cup B = \{2, 5, 3, 7\}$
   $A \cap B = \{3\}$

9. $A \cup B = \{64, 81, 100, 121, 144\}$
   $A \cap B = \{64, 81\}$

11. Disjoint sets.

13. $A' = \{5, 6, 7, 8\}$
    $B' = \{2, 4, 5, 6, 8\}$

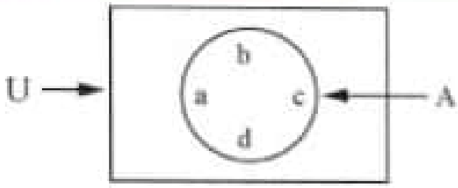
15. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$

17. (a) $A' \cap B = \{5, 6\}$ (b) $A \cap B' = \{1, 2\}$
    (c) $B' \cup A' = \{1, 2, 5, 6, 7\}$ (d) $B \cup A' = \{3, 4, 5, 6, 7\}$

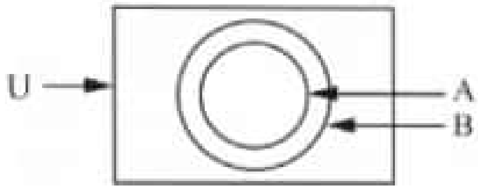
19. (a) $A \cup F = \{2, 4, 6, 8, 10\}$
    $A \cap F = \{4, 6\}$
    (b) $Y \cup W = \{x : 2 < x < 11\}$
    $Y \cap W = \{x : 7 \leq x \leq 8\}$

Exercise 10.5

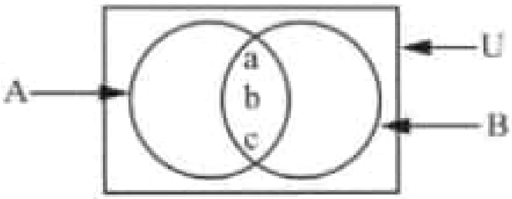
1. (a)



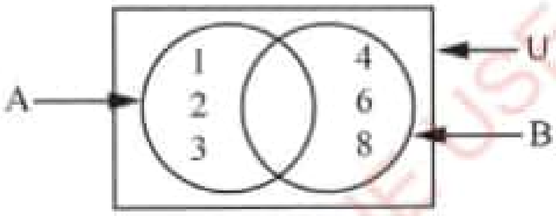
(b)



(c)



(d)

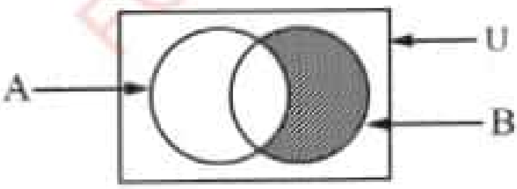


3. $A \cap B$ 5. $n(A) = 0$

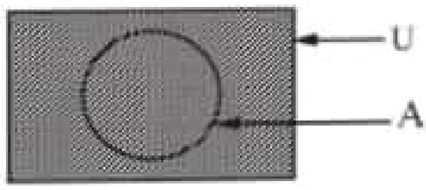
7. (a) $n(A) = 10$ (b) $B \subset A$

Exercise 10.6

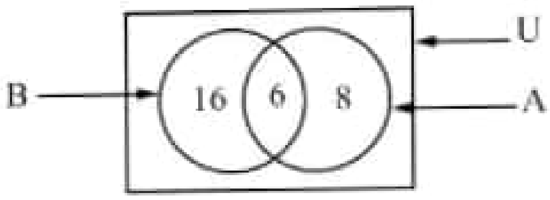
1.



3.



5.



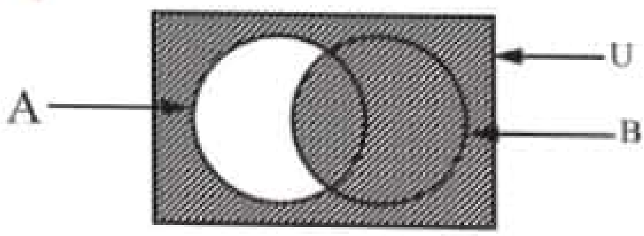
Therefore, from the Venn diagram n(B) = 16 + 6 = 22

Exercise 10.7

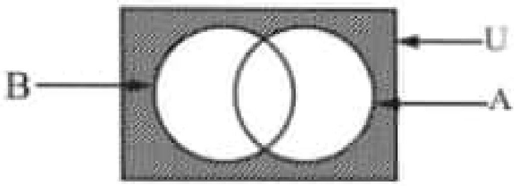
1. 20 students
3. 7 students
5. 14 students
7. 32 women
9. 52 men
11. 15 students

Revision exercise 10

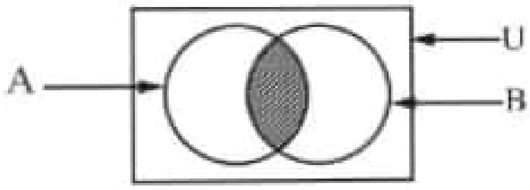
1. (c) and (d)
3. A = {squares of the first five counting numbers}
5. (c) and (d)
7. B, C and E are finite sets (b) A and C are infinite sets
9. 128
11. (a) $A \subset B$ (b) $D \subset C$
13. (a) $A \cup B = \{a, b, c, d\}$ (b) $A \cap B = \{\ \}$
15. $A = \{1, 4\}$
17. (a) $A' \cup B$



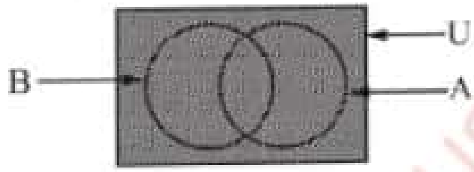
(b) A′∪B



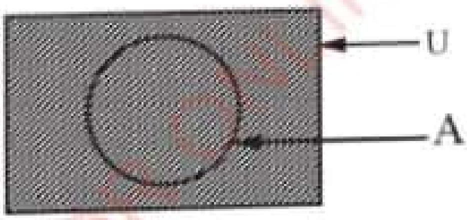
(c) A∩B



(d) U∪(A′∪B)



19.

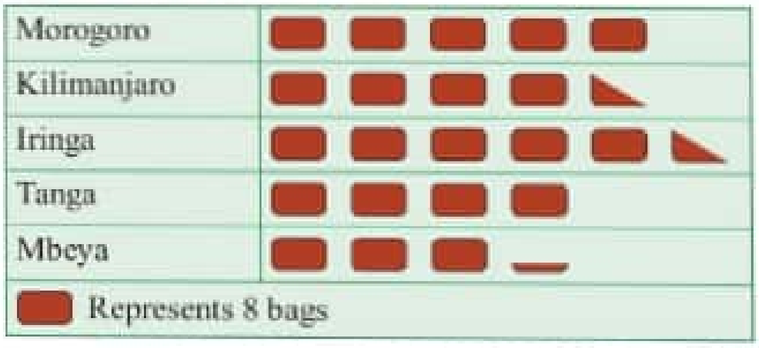


21. 115 houses

CHAPTER 11

Exercises 11.1

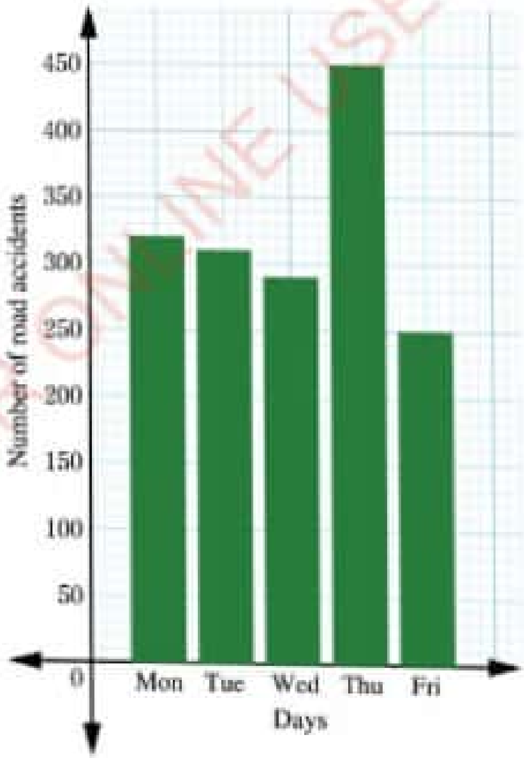
1.



3. (a) 20% (b) 65% (c) 50% (d) 24 students (e) 45%

Exercise 11.2

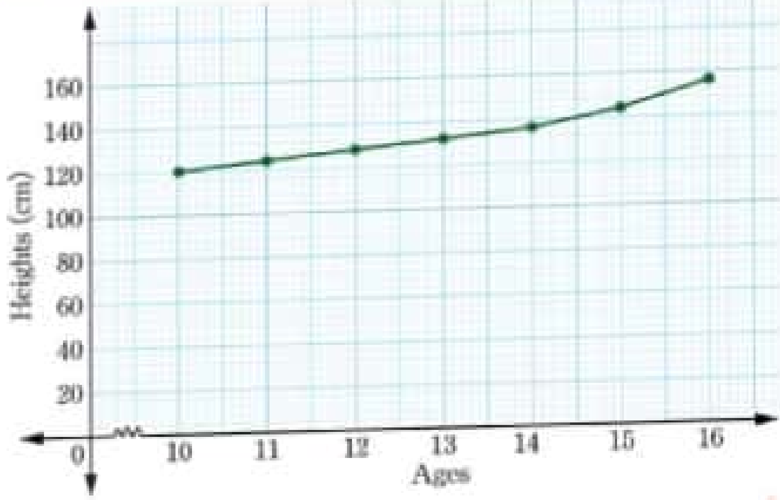
1.



3. (a) 2 (b) 145 cm (c) 14

Exercise 11.3

1.



3. (a) 10 years (b) 25 kg (c) 45 kg

5. (a)

(b) 38.9^0

(c) at 11.00 am

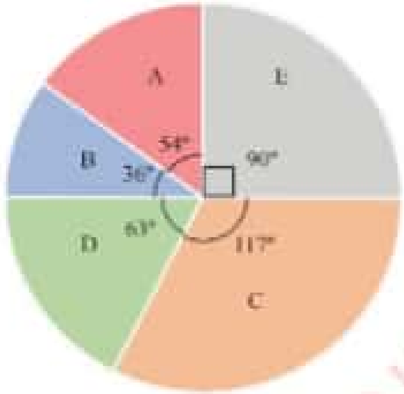
Exercise 11.4

1.

3.

Exercise 11.5

1.

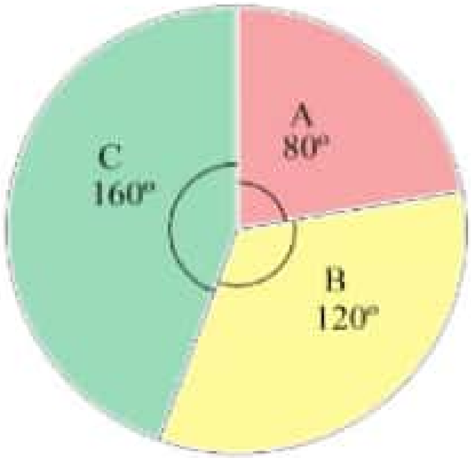


3. (a)

(b) 5.2 kg

5. (a) A = 80°, B = 120°, C = 160°

(b)



Exercise 11.6

1. Frequency distribution table:

Class interval	Frequency
20 – 29	0
30 – 39	7
40 – 49	10
50 – 59	8
60 – 69	2
70 – 79	3

3. Frequency distribution table:

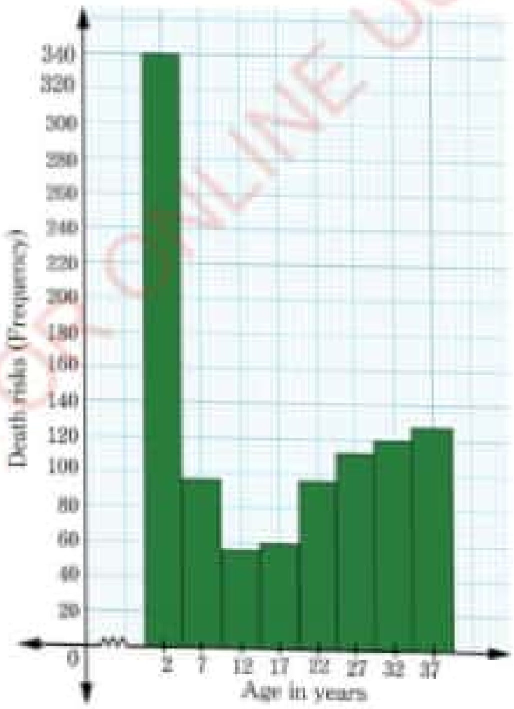
Class interval	Frequency
50 – 54	3
55 – 59	2
60 – 64	7
65 – 69	5
70 – 74	10
75 – 79	3
80 – 84	4
85 – 89	1
90 – 94	1
95 – 99	0

5. Frequency distribution table:

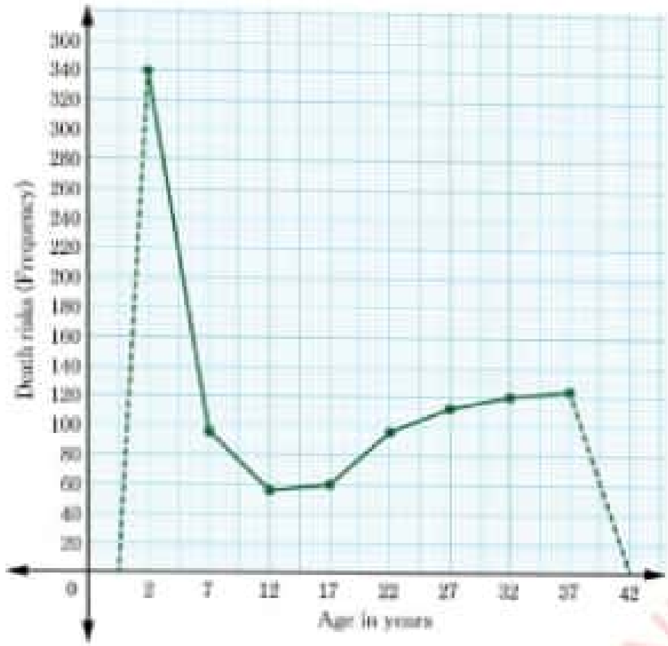
Class interval	Class Mark	Interval size	Real limits	Frequency
8 – 15	11.5	8	7.5 – 15.5	4
16 – 23	19.5	8	15.5 – 23.5	2
24 – 31	27.5	8	23.5 – 31.5	8
32 – 39	35.5	8	31.5 – 39.5	4
40 – 47	43.5	8	39.5 – 47.5	7
48 – 55	51.5	8	47.5 – 55.5	4
56 – 63	59.5	8	55.5 – 63.5	6
64 – 71	67.5	8	63.5 – 71.5	4
72 – 79	75.5	8	71.5 – 79.5	5
80 – 87	83.5	8	79.5 – 87.5	3
88 – 95	91.5	8	87.5 – 95.5	3
96 – 103	99.5	8	95.5 – 103.5	0

Exercise 11.7

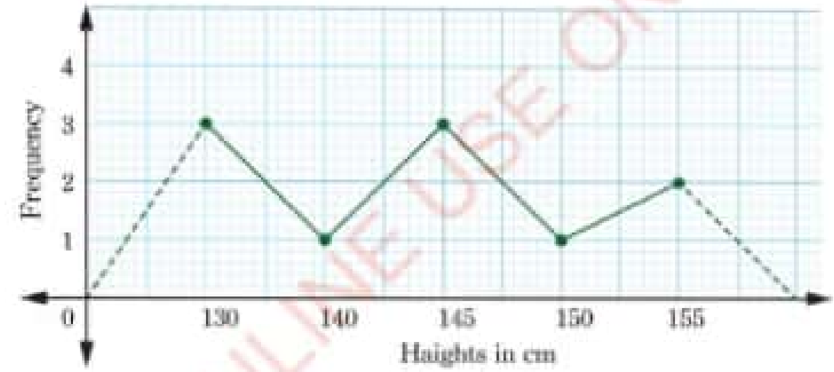
1. (a) Histogram



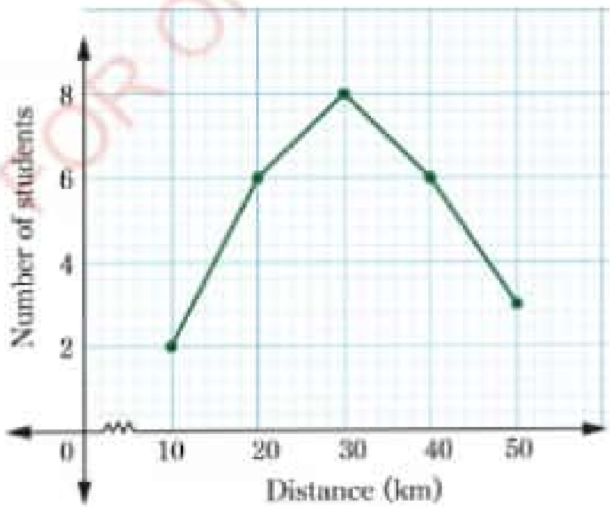
(b) A frequency polygon:



3.



5.



7. (a) True

(b) False

Exercise 11.8

1.

Number of beds	Frequency	Cumulative frequency
70 – 79	5	50
60 – 69	5	45
50 – 59	1	40
40 – 49	12	39
30 – 39	12	27
20 – 29	15	15

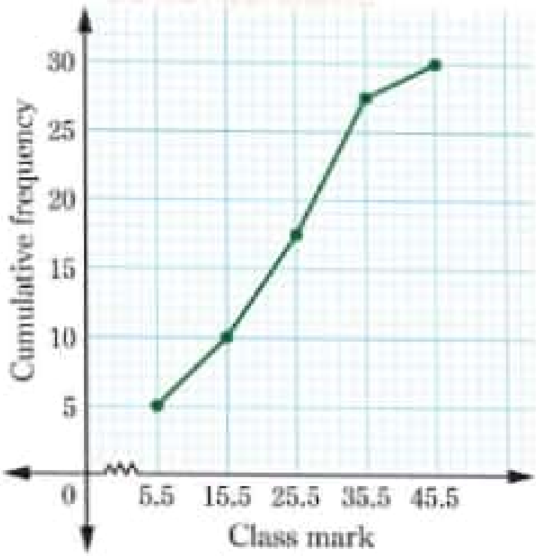
3. (a)

Goals	Frequency	Cumulative frequency
11	1	100
10	0	99
9	0	99
8	2	99
7	5	97
6	7	92
5	10	85
4	16	75
3	22	59
2	17	37
1	15	20
0	5	5

(b) 92 matches (c) 15 matches (d) 17 matches

5.

Class interval	Frequency	Cumulative frequency
1 – 10	4	4
11 – 20	6	10
21 – 30	8	18
31 – 40	9	27
41 – 50	3	30



Revision exercise 11

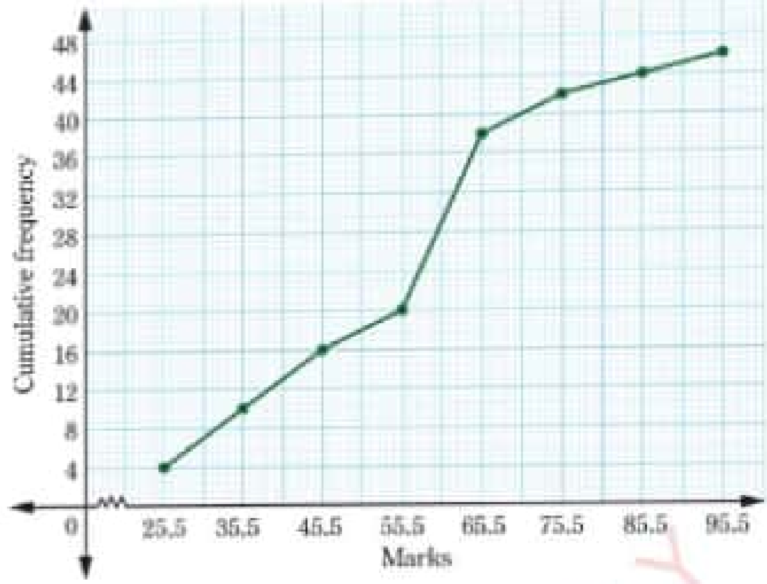
1. Frequency table

Ages	Frequency
15	10
16	5
17	2
18	5

3. (a) Frequency distribution

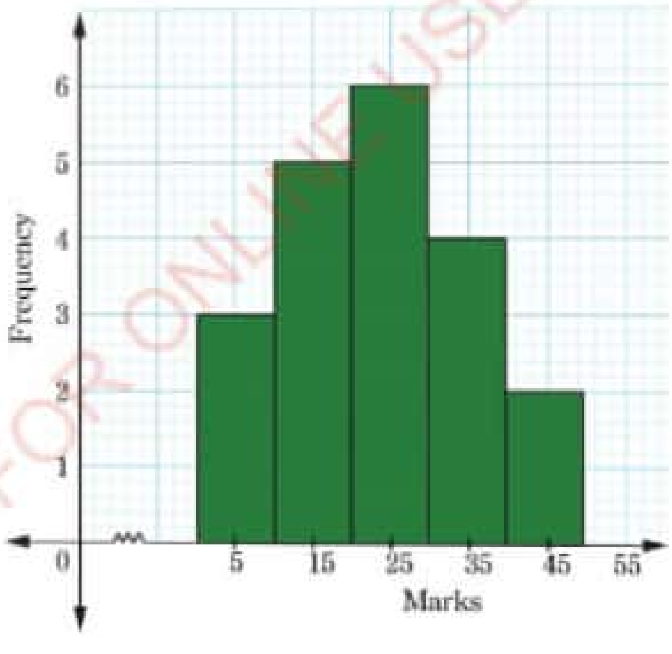
Class interval	x	f	Cumulative frequency
21 – 30	25.5	4	4
31 – 40	35.5	6	10
41 – 50	45.5	6	16
51 – 60	55.5	4	20
61 – 70	65.5	14	34
71 – 80	75.5	9	43
81 – 90	85.5	1	44
91 – 100	95.5	1	45

(b)



Cumulative frequency
48
44
40
36
32
28
24
20
16
12
8
4
0
25.5
35.5
45.5
55.5
65.5
75.5
85.5
95.5
Marks

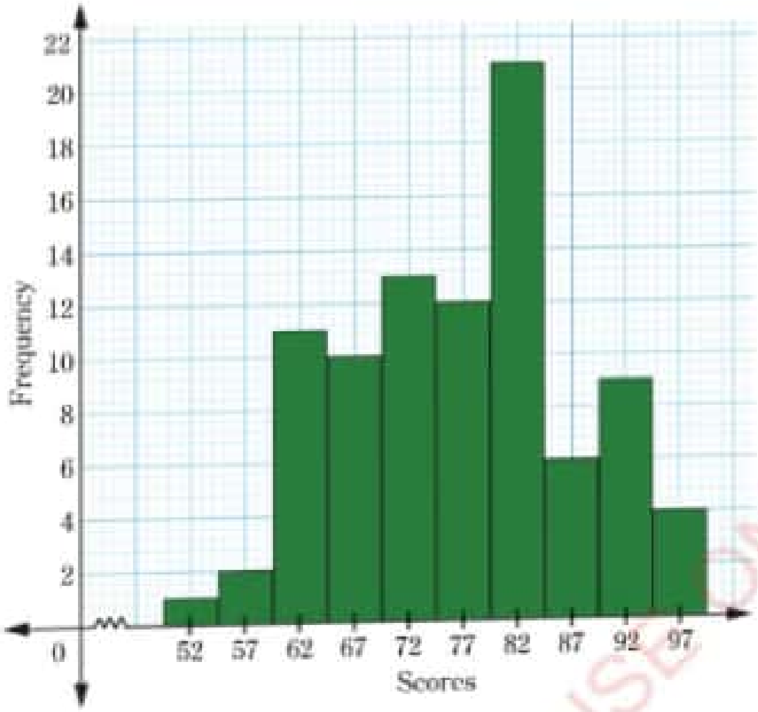
5.



Frequency
6
5
4
3
2
1
0
5
15
25
35
45
55
Marks

7. (a) Size of class interval is 5

(b)



Mathematical Tables

Common Logarithms of Numbers

$\log_{10} x$ or $\log x$

x	0	1	2	3	4	5	6	7	8	9	Mean Differences (Add) 1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170	0.0212	0.0253	0.0294	0.0334	0.0374	4	8	12	17	21	25	29	33	37
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755	4	8	11	15	19	23	26	30	34
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106	3	7	10	14	17	21	24	28	31
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430	3	6	10	13	16	19	23	26	29
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732	3	6	9	12	15	18	21	24	27
1.5	0.1761	0.1790	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014	3	6	8	11	14	17	20	22	25
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279	3	5	8	11	13	16	18	21	24
1.7	0.2304	0.2330	0.2355	0.2380	0.2405	0.2430	0.2455	0.2480	0.2504	0.2529	2	5	7	10	12	15	17	20	22
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765	2	5	7	9	12	14	16	19	21
1.9	0.2788	0.2810	0.2833	0.2856	0.2878	0.2900	0.2923	0.2945	0.2967	0.2989	2	4	7	9	11	13	16	18	20
2.0	0.3010	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.3160	0.3181	0.3201	2	4	6	8	11	13	15	17	19
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404	2	4	6	8	10	12	14	16	18
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.3560	0.3579	0.3598	2	4	6	8	10	12	14	15	17
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784	2	4	6	7	9	11	13	15	17
2.4	0.3802	0.3820	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962	2	4	5	7	9	11	12	14	16
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133	2	3	5	7	9	10	12	14	15
2.6	0.4150	0.4166	0.4183	0.4200	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298	2	3	5	7	8	10	11	13	15
2.7	0.4314	0.4330	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.4440	0.4456	2	3	5	6	8	9	11	13	14
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609	2	3	5	6	8	9	11	12	14
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757	1	3	4	6	7	9	10	12	13
3.0	0.4771	0.4786	0.4800	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.4900	1	3	4	6	7	9	10	11	13
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038	1	3	4	6	7	8	10	11	12
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172	1	3	4	5	7	8	9	11	12
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.5250	0.5263	0.5276	0.5289	0.5302	1	3	4	5	6	8	9	10	12
3.4	0.5315	0.5328	0.5340	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428	1	3	4	5	6	8	9	10	11
3.5	0.5441	0.5453	0.5465	0.5478	0.5490	0.5502	0.5514	0.5527	0.5539	0.5551	1	2	4	5	6	7	9	10	11
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.5670	1	2	4	5	6	7	8	10	11
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.5740	0.5752	0.5763	0.5775	0.5786	1	2	3	5	6	7	8	9	10
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899	1	2	3	5	6	7	8	9	10
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.6010	1	2	3	4	6	7	8	9	10
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117	1	2	3	4	5	6	8	9	10
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

$\log_{10} x$ or $\log x$

x	0	1	2	3	4	5	6	7	8	9	Mean Differences (Add) 1	2	3	4	5	6	7	8	9
4.1	0.6128	0.6138	0.6149	0.6160	0.6170	0.6180	0.6191	0.6201	0.6212	0.6222	1	2	3	4	5	6	7	8	9
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325	1	2	3	4	5	6	7	8	9
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425	1	2	3	4	5	6	7	8	9
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522	1	2	3	4	5	6	7	8	9
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.6580	0.6590	0.6599	0.6609	0.6618	1	2	3	4	5	6	7	8	9
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712	1	2	3	4	5	6	7	7	8
4.7	0.6721	0.6730	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803	1	2	3	4	5	5	6	7	8
4.8	0.6812	0.6821	0.6830	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893	1	2	3	4	4	5	6	7	8
4.9	0.6902	0.6911	0.6920	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981	1	2	3	4	4	5	6	7	8
5.0	0.6990	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.7050	0.7059	0.7067	1	2	3	3	4	5	6	7	8
5.1	0.7076	0.7084	0.7093	0.7101	0.7110	0.7118	0.7126	0.7135	0.7143	0.7152	1	2	3	3	4	5	6	7	8
5.2	0.7160	0.7168	0.7177	0.7185	0.7193	0.7202	0.7210	0.7218	0.7226	0.7235	1	2	2	3	4	5	6	7	7
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.7300	0.7308	0.7316	1	2	2	3	4	5	6	6	7
5.4	0.7324	0.7332	0.7340	0.7348	0.7356	0.7364	0.7372	0.7380	0.7388	0.7396	1	2	2	3	4	5	6	6	7
5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474	1	2	2	3	4	5	5	6	7
5.6	0.7482	0.7490	0.7497	0.7505	0.7513	0.7520	0.7528	0.7536	0.7543	0.7551	1	2	2	3	4	5	5	6	7
5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627	1	2	2	3	4	5	5	6	7
5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701	1	1	2	3	4	4	5	6	7
5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.7760	0.7767	0.7774	1	1	2	3	4	4	5	6	7
6.0	0.7782	0.7789	0.7796	0.7803	0.7810	0.7818	0.7825	0.7832	0.7839	0.7846	1	1	2	3	4	4	5	6	6
6.1	0.7853	0.7860	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.7910	0.7917	1	1	2	3	4	4	5	6	6
6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.7980	0.7987	1	1	2	3	3	4	5	6	6
6.3	0.7993	0.8000	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055	1	1	2	3	3	4	5	5	6
6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122	1	1	2	3	3	4	5	5	6
6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189	1	1	2	3	3	4	5	5	6
6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254	1	1	2	3	3	4	5	5	6
6.7	0.8261	0.8267	0.8274	0.8280	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319	1	1	2	3	3	4	5	5	6
6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.8370	0.8376	0.8382	1	1	2	3	3	4	4	5	6
6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.8420	0.8426	0.8432	0.8439	0.8445	1	1	2	3	3	4	4	5	6
7.0	0.8451	0.8457	0.8463	0.8470	0.8476	0.8482	0.8488	0.8494	0.8500	0.8506	1	1	2	2	3	4	4	5	6
7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567	1	1	2	2	3	4	4	5	5
7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627	1	1	2	2	3	4	4	5	5
7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686	1	1	2	2	3	4	4	5	5
7.4	0.8692	0.8698	0.8704	0.8710	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745	1	1	2	2	3	3	4	5	5
7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802	1	1	2	2	3	3	4	5	5
7.6	0.8808	0.8814	0.8820	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859	1	1	2	2	3	3	4	5	5
7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.8910	0.8915	1	1	2	2	3	3	4	4	5
7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.8960	0.8965	0.8971	1	1	2	2	3	3	4	4	5
7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.9020	0.9025	1	1	2	2	3	3	4	4	5
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

$\log_{10} x$ or $\log x$

x	0	1	2	3	4	5	6	7	8	9	Mean Differences (Add) 1	2	3	4	5	6	7	8	9
8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079	1	1	2	2	3	3	4	4	5
8.1	0.9085	0.9090	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133	1	1	2	2	3	3	4	4	5
8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.9170	0.9175	0.9180	0.9186	1	1	2	2	3	3	4	4	5
8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238	1	1	2	2	3	3	4	4	5
8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289	1	1	2	2	3	3	4	4	5
8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.9320	0.9325	0.9330	0.9335	0.9340	1	1	2	2	3	3	4	4	5
8.6	0.9345	0.9350	0.9355	0.9360	0.9365	0.9370	0.9375	0.9380	0.9385	0.9390	1	1	2	2	3	3	4	4	5
8.7	0.9395	0.9400	0.9405	0.9410	0.9415	0.9420	0.9425	0.9430	0.9435	0.9440	0	1	1	2	2	3	3	4	4
8.8	0.9445	0.9450	0.9455	0.9460	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489	0	1	1	2	2	3	3	4	4
8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538	0	1	1	2	2	3	3	4	4
9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586	0	1	1	2	2	3	3	4	4
9.1	0.9590	0.9595	0.9600	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633	0	1	1	2	2	3	3	4	4
9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.9680	0	1	1	2	2	3	3	4	4
9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727	0	1	1	2	2	3	3	4	4
9.4	0.9731	0.9736	0.9741	0.9745	0.9750	0.9754	0.9759	0.9763	0.9768	0.9773	0	1	1	2	2	3	3	4	4
9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.9800	0.9805	0.9809	0.9814	0.9818	0	1	1	2	2	3	3	4	4
9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.9850	0.9854	0.9859	0.9863	0	1	1	2	2	3	3	4	4
9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	0.9908	0	1	1	2	2	3	3	4	4
9.8	0.9912	0.9917	0.9921	0.9926	0.9930	0.9934	0.9939	0.9943	0.9948	0.9952	0	1	1	2	2	3	3	4	4
9.9	0.9956	0.9961	0.9965	0.9969	0.9974	0.9978	0.9983	0.9987	0.9991	0.9996	0	1	1	2	2	3	3	3	4
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Example: $\log_{10} 9.592 = 0.9819$

Antilogarithms

10^x

x	0	1	2	3	4	5	6	7	8	9	Mean Differences (Add) 1	2	3	4	5	6	7	8	9
.00	1.000	1.002	1.005	1.007	1.009	1.012	1.014	1.016	1.019	1.021									
.01	1.023	1.026	1.028	1.030	1.033	1.035	1.038	1.040	1.042	1.045	0	0	1	1	1	1	2	2	2
.02	1.047	1.050	1.052	1.054	1.057	1.059	1.062	1.064	1.067	1.069	0	0	1	1	1	1	2	2	2
.03	1.072	1.074	1.077	1.079	1.081	1.084	1.086	1.089	1.091	1.094	0	0	1	1	1	1	2	2	2
.04	1.097	1.099	1.102	1.104	1.107	1.109	1.112	1.114	1.117	1.119	0	0	1	1	1	1	2	2	2
.05	1.122	1.125	1.127	1.130	1.132	1.135	1.138	1.140	1.143	1.146	0	1	1	1	1	2	2	2	2
.06	1.148	1.151	1.154	1.156	1.159	1.161	1.164	1.167	1.170	1.172	0	1	1	1	1	2	2	2	2
.07	1.175	1.178	1.180	1.183	1.186	1.189	1.191	1.194	1.197	1.200	0	1	1	1	1	2	2	2	2
.08	1.202	1.205	1.208	1.211	1.213	1.216	1.219	1.222	1.225	1.227	0	1	1	1	1	2	2	2	2
.09	1.230	1.233	1.236	1.239	1.242	1.245	1.247	1.250	1.253	1.256	0	1	1	1	1	2	2	2	3
.10	1.259	1.262	1.265	1.268	1.271	1.274	1.276	1.279	1.282	1.285	0	1	1	1	1	2	2	2	3
.11	1.288	1.291	1.294	1.297	1.300	1.303	1.306	1.309	1.312	1.315	0	1	1	1	1	2	2	2	3
.12	1.318	1.321	1.324	1.327	1.331	1.334	1.337	1.340	1.343	1.346	0	1	1	1	2	2	2	2	3
.13	1.349	1.352	1.355	1.358	1.361	1.365	1.368	1.371	1.374	1.377	0	1	1	1	2	2	2	2	3
.14	1.380	1.384	1.387	1.390	1.393	1.396	1.400	1.403	1.406	1.409	0	1	1	1	2	2	2	3	3
.15	1.413	1.416	1.419	1.422	1.426	1.429	1.432	1.436	1.439	1.442	0	1	1	1	2	2	2	3	3
.16	1.445	1.449	1.452	1.456	1.459	1.462	1.466	1.469	1.472	1.476	0	1	1	1	2	2	2	3	3
.17	1.479	1.483	1.486	1.489	1.493	1.496	1.500	1.503	1.507	1.510	0	1	1	1	2	2	2	3	3
.18	1.514	1.517	1.521	1.524	1.528	1.531	1.535	1.538	1.542	1.545	0	1	1	1	2	2	2	3	3
.19	1.549	1.552	1.556	1.560	1.563	1.567	1.570	1.574	1.578	1.581	0	1	1	1	2	2	3	3	3
.20	1.585	1.589	1.592	1.596	1.600	1.603	1.607	1.611	1.614	1.618	0	1	1	1	2	2	3	3	3
.21	1.622	1.626	1.629	1.633	1.637	1.641	1.644	1.648	1.652	1.656	0	1	1	2	2	2	3	3	3
.22	1.660	1.663	1.667	1.671	1.675	1.679	1.683	1.687	1.690	1.694	0	1	1	2	2	2	3	3	3
.23	1.698	1.702	1.706	1.710	1.714	1.718	1.722	1.726	1.730	1.734	0	1	1	2	2	2	3	3	4
.24	1.738	1.742	1.746	1.750	1.754	1.758	1.762	1.766	1.770	1.774	0	1	1	2	2	2	3	3	4
.25	1.778	1.782	1.787	1.791	1.795	1.799	1.803	1.807	1.811	1.816	0	1	1	2	2	2	3	3	4
.26	1.820	1.824	1.828	1.832	1.837	1.841	1.845	1.849	1.854	1.858	0	1	1	2	2	3	3	3	4
.27	1.862	1.866	1.871	1.875	1.879	1.884	1.888	1.892	1.897	1.901	0	1	1	2	2	3	3	3	4
.28	1.906	1.910	1.914	1.919	1.923	1.928	1.932	1.936	1.941	1.945	0	1	1	2	2	3	3	4	4
.29	1.950	1.954	1.959	1.963	1.968	1.972	1.977	1.982	1.986	1.991	0	1	1	2	2	3	3	4	4
.30	1.995	2.000	2.005	2.009	2.014	2.018	2.023	2.028	2.032	2.037	0	1	1	2	2	3	3	4	4
.31	2.042	2.046	2.051	2.056	2.061	2.065	2.070	2.075	2.080	2.085	0	1	1	2	2	3	3	4	4
.32	2.089	2.094	2.099	2.104	2.109	2.114	2.118	2.123	2.128	2.133	0	1	1	2	2	3	3	4	4
.33	2.138	2.143	2.148	2.153	2.158	2.163	2.168	2.173	2.178	2.183	0	1	1	2	2	3	3	4	4
.34	2.188	2.193	2.198	2.203	2.208	2.213	2.218	2.223	2.228	2.234	1	1	2	2	3	3	4	4	5
.35	2.239	2.244	2.249	2.254	2.259	2.265	2.270	2.275	2.280	2.286	1	1	2	2	3	3	4	4	5
.36	2.291	2.296	2.301	2.307	2.312	2.317	2.323	2.328	2.334	2.339	1	1	2	2	3	3	4	4	5
.37	2.344	2.350	2.355	2.361	2.366	2.371	2.377	2.382	2.388	2.393	1	1	2	2	3	3	4	4	5
.38	2.399	2.404	2.410	2.416	2.421	2.427	2.432	2.438	2.443	2.449	1	1	2	2	3	3	4	4	5
.39	2.455	2.460	2.466	2.472	2.477	2.483	2.489	2.495	2.500	2.506	1	1	2	2	3	3	4	5	5
.40	2.512	2.518	2.524	2.529	2.535	2.541	2.547	2.553	2.559	2.565	1	1	2	2	3	4	4	5	5
.41	2.570	2.576	2.582	2.588	2.594	2.600	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
.42	2.630	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
.43	2.692	2.698	2.704	2.710	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
.44	2.754	2.761	2.767	2.773	2.780	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6
.45	2.818	2.825	2.831	2.838	2.845	2.851	2.858	2.864	2.871	2.877	1	1	2	3	3	4	5	5	6
.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1	1	2	3	3	4	5	5	6
.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1	1	2	3	3	4	5	5	6
.48	3.020	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1	1	2	3	4	4	5	6	6
.49	3.090	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1	1	2	3	4	4	5	6	6
.50	3.162	3.170	3.177	3.184	3.192	3.199	3.206	3.214	3.221	3.229	1	1	2	3	4	4	5	6	7
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

10^x

x	0	1	2	3	4	5	6	7	8	9	Mean Differences (Add)								
											1	2	3	4	5	6	7	8	9
.51	3.236	3.243	3.251	3.258	3.266	3.273	3.281	3.289	3.296	3.304	1	2	2	3	4	5	5	6	7
.52	3.311	3.319	3.327	3.334	3.342	3.350	3.357	3.365	3.373	3.381	1	2	2	3	4	5	5	6	7
.53	3.388	3.396	3.404	3.412	3.420	3.428	3.436	3.444	3.451	3.459	1	2	2	3	4	5	6	6	7
.54	3.467	3.475	3.483	3.491	3.500	3.508	3.516	3.524	3.532	3.540	1	2	2	3	4	5	6	6	7
.55	3.548	3.556	3.565	3.573	3.581	3.589	3.598	3.606	3.614	3.622	1	2	2	3	4	5	6	7	7
.56	3.631	3.639	3.648	3.656	3.664	3.673	3.681	3.690	3.698	3.707	1	2	3	3	4	5	6	7	8
.57	3.715	3.724	3.733	3.741	3.750	3.758	3.767	3.776	3.784	3.793	1	2	3	3	4	5	6	7	8
.58	3.802	3.811	3.819	3.828	3.837	3.846	3.855	3.864	3.873	3.882	1	2	3	4	4	5	6	7	8
.59	3.891	3.899	3.908	3.917	3.926	3.936	3.945	3.954	3.963	3.972	1	2	3	4	5	5	6	7	8
.60	3.981	3.990	3.999	4.009	4.018	4.027	4.037	4.046	4.055	4.064	1	2	3	4	5	6	6	7	8
.61	4.074	4.083	4.093	4.102	4.112	4.121	4.131	4.140	4.150	4.159	1	2	3	4	5	6	7	8	9
.62	4.169	4.178	4.188	4.198	4.207	4.217	4.227	4.236	4.246	4.256	1	2	3	4	5	6	7	8	9
.63	4.266	4.276	4.286	4.295	4.305	4.315	4.325	4.335	4.345	4.355	1	2	3	4	5	6	7	8	9
.64	4.365	4.375	4.385	4.395	4.406	4.416	4.426	4.436	4.446	4.457	1	2	3	4	5	6	7	8	9
.65	4.467	4.477	4.488	4.498	4.508	4.519	4.529	4.539	4.550	4.560	1	2	3	4	5	6	7	8	9
.66	4.571	4.581	4.592	4.603	4.613	4.624	4.635	4.645	4.656	4.667	1	2	3	4	5	6	7	9	10
.67	4.677	4.688	4.699	4.710	4.721	4.732	4.742	4.753	4.764	4.775	1	2	3	4	5	7	8	9	10
.68	4.786	4.797	4.808	4.820	4.831	4.842	4.853	4.864	4.875	4.887	1	2	3	4	6	7	8	9	10
.69	4.898	4.909	4.920	4.932	4.943	4.955	4.966	4.977	4.989	5.000	1	2	3	5	6	7	8	9	10
.70	5.012	5.023	5.035	5.047	5.058	5.070	5.082	5.093	5.105	5.117	1	2	3	5	6	7	8	9	11
.71	5.129	5.140	5.152	5.164	5.176	5.188	5.200	5.212	5.224	5.236	1	2	4	5	6	7	8	10	11
.72	5.248	5.260	5.272	5.285	5.297	5.309	5.321	5.333	5.346	5.358	1	2	4	5	6	7	9	10	11
.73	5.370	5.383	5.395	5.408	5.420	5.433	5.445	5.458	5.470	5.483	1	2	4	5	6	8	9	10	11
.74	5.495	5.508	5.521	5.534	5.546	5.559	5.572	5.585	5.598	5.611	1	3	4	5	6	8	9	10	12
.75	5.623	5.636	5.649	5.662	5.675	5.689	5.702	5.715	5.728	5.741	1	3	4	5	7	8	9	10	12
.76	5.754	5.768	5.781	5.794	5.808	5.821	5.835	5.848	5.861	5.875	1	3	4	5	7	8	9	11	12
.77	5.888	5.902	5.916	5.929	5.943	5.957	5.970	5.984	5.998	6.012	1	3	4	5	7	8	10	11	12
.78	6.026	6.040	6.053	6.067	6.081	6.095	6.109	6.124	6.138	6.152	1	3	4	6	7	8	10	11	13
.79	6.166	6.180	6.194	6.209	6.223	6.237	6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
.80	6.310	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1	3	4	6	7	9	10	12	13
.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.562	6.577	6.592	2	3	5	6	8	9	11	12	14
.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.730	6.745	2	3	5	6	8	9	11	12	14
.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2	3	5	6	8	9	11	13	14
.84	6.918	6.934	6.950	6.966	6.982	6.998	7.015	7.031	7.047	7.063	2	3	5	6	8	10	11	13	15
.85	7.080	7.096	7.112	7.129	7.145	7.161	7.178	7.195	7.211	7.228	2	3	5	7	8	10	12	13	15
.86	7.244	7.261	7.278	7.295	7.311	7.328	7.345	7.362	7.379	7.396	2	3	5	7	8	10	12	13	15
.87	7.413	7.430	7.447	7.465	7.482	7.499	7.516	7.534	7.551	7.568	2	3	5	7	9	10	12	14	16
.88	7.586	7.603	7.621	7.638	7.656	7.674	7.691	7.709	7.727	7.745	2	4	5	7	9	11	12	14	16
.89	7.763	7.780	7.798	7.816	7.834	7.852	7.871	7.889	7.907	7.925	2	4	5	7	9	11	13	14	16
.90	7.943	7.962	7.980	7.998	8.017	8.035	8.054	8.072	8.091	8.110	2	4	6	7	9	11	13	15	17
.91	8.128	8.147	8.166	8.185	8.204	8.222	8.241	8.260	8.279	8.299	2	4	6	8	9	11	13	15	17
.92	8.318	8.337	8.356	8.375	8.395	8.414	8.433	8.453	8.472	8.492	2	4	6	8	10	12	14	15	17
.93	8.511	8.531	8.551	8.570	8.590	8.610	8.630	8.650	8.670	8.690	2	4	6	8	10	12	14	16	18
.94	8.710	8.730	8.750	8.770	8.790	8.811	8.831	8.851	8.872	8.892	2	4	6	8	10	12	14	16	18
.95	8.913	8.933	8.954	8.974	8.995	9.016	9.037	9.057	9.078	9.099	2	4	6	8	10	12	15	17	19
.96	9.120	9.141	9.162	9.183	9.205	9.226	9.247	9.268	9.290	9.311	2	4	6	8	11	13	15	17	19
.97	9.333	9.354	9.376	9.397	9.419	9.441	9.462	9.484	9.506	9.528	2	4	7	9	11	13	15	17	20
.98	9.550	9.572	9.594	9.616	9.638	9.661	9.683	9.705	9.728	9.750	2	4	7	9	11	13	16	18	20
.99	9.772	9.795	9.818	9.840	9.863	9.886	9.908	9.931	9.954	9.977	2	5	7	9	11	14	16	18	21
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Example: *Anti* log 0.8523 = $10^{0.8523}$ = 7.117

Sine of an Angle

sin x

x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences (Add)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332	3	6	9	12	15
2	0.0349	0.0366	0.0384	0.0401	0.0419	0.0436	0.0454	0.0471	0.0488	0.0506	3	6	9	12	15
3	0.0523	0.0541	0.0558	0.0576	0.0593	0.0610	0.0628	0.0645	0.0663	0.0680	3	6	9	12	15
4	0.0698	0.0715	0.0732	0.0750	0.0767	0.0785	0.0802	0.0819	0.0837	0.0854	3	6	9	12	14
5	0.0872	0.0889	0.0906	0.0924	0.0941	0.0958	0.0976	0.0993	0.1011	0.1028	3	6	9	12	14
6	0.1045	0.1063	0.1080	0.1097	0.1115	0.1132	0.1149	0.1167	0.1184	0.1201	3	6	9	12	14
7	0.1219	0.1236	0.1253	0.1271	0.1288	0.1305	0.1323	0.1340	0.1357	0.1374	3	6	9	12	14
8	0.1392	0.1409	0.1426	0.1444	0.1461	0.1478	0.1495	0.1513	0.1530	0.1547	3	6	9	12	14
9	0.1564	0.1582	0.1599	0.1616	0.1633	0.1650	0.1668	0.1685	0.1702	0.1719	3	6	9	12	14
10	0.1736	0.1754	0.1771	0.1788	0.1805	0.1822	0.1840	0.1857	0.1874	0.1891	3	6	9	11	14
11	0.1908	0.1925	0.1942	0.1959	0.1977	0.1994	0.2011	0.2028	0.2045	0.2062	3	6	9	11	14
12	0.2079	0.2096	0.2113	0.2130	0.2147	0.2164	0.2181	0.2198	0.2215	0.2233	3	6	9	11	14
13	0.2250	0.2267	0.2284	0.2300	0.2317	0.2334	0.2351	0.2368	0.2385	0.2402	3	6	8	11	14
14	0.2419	0.2436	0.2453	0.2470	0.2487	0.2504	0.2521	0.2538	0.2554	0.2571	3	6	8	11	14
15	0.2588	0.2605	0.2622	0.2639	0.2656	0.2672	0.2689	0.2706	0.2723	0.2740	3	6	8	11	14
16	0.2756	0.2773	0.2790	0.2807	0.2823	0.2840	0.2857	0.2874	0.2890	0.2907	3	6	8	11	14
17	0.2924	0.2940	0.2957	0.2974	0.2990	0.3007	0.3024	0.3040	0.3057	0.3074	3	6	8	11	14
18	0.3090	0.3107	0.3123	0.3140	0.3156	0.3173	0.3190	0.3206	0.3223	0.3239	3	6	8	11	14
19	0.3256	0.3272	0.3289	0.3305	0.3322	0.3338	0.3355	0.3371	0.3387	0.3404	3	5	8	11	14
20	0.3420	0.3437	0.3453	0.3469	0.3486	0.3502	0.3518	0.3535	0.3551	0.3567	3	5	8	11	14
21	0.3584	0.3600	0.3616	0.3633	0.3649	0.3665	0.3681	0.3697	0.3714	0.3730	3	5	8	11	14
22	0.3746	0.3762	0.3778	0.3795	0.3811	0.3827	0.3843	0.3859	0.3875	0.3891	3	5	8	11	14
23	0.3907	0.3923	0.3939	0.3955	0.3971	0.3987	0.4003	0.4019	0.4035	0.4051	3	5	8	11	14
24	0.4067	0.4083	0.4099	0.4115	0.4131	0.4147	0.4163	0.4179	0.4195	0.4210	3	5	8	11	13
25	0.4226	0.4242	0.4258	0.4274	0.4289	0.4305	0.4321	0.4337	0.4352	0.4368	3	5	8	11	13
26	0.4384	0.4399	0.4415	0.4431	0.4446	0.4462	0.4478	0.4493	0.4509	0.4524	3	5	8	10	13
27	0.4540	0.4555	0.4571	0.4586	0.4602	0.4617	0.4633	0.4648	0.4664	0.4679	3	5	8	10	13
28	0.4695	0.4710	0.4726	0.4741	0.4756	0.4772	0.4787	0.4802	0.4818	0.4833	3	5	8	10	13
29	0.4848	0.4863	0.4879	0.4894	0.4909	0.4924	0.4939	0.4955	0.4970	0.4985	3	5	8	10	13
30	0.5000	0.5015	0.5030	0.5045	0.5060	0.5075	0.5090	0.5105	0.5120	0.5135	3	5	8	10	13
31	0.5150	0.5165	0.5180	0.5195	0.5210	0.5225	0.5240	0.5255	0.5270	0.5284	2	5	7	10	12
32	0.5299	0.5314	0.5329	0.5344	0.5358	0.5373	0.5388	0.5402	0.5417	0.5432	2	5	7	10	12
33	0.5446	0.5461	0.5476	0.5490	0.5505	0.5519	0.5534	0.5548	0.5563	0.5577	2	5	7	10	12
34	0.5592	0.5606	0.5621	0.5635	0.5650	0.5664	0.5678	0.5693	0.5707	0.5721	2	5	7	10	12
35	0.5736	0.5750	0.5764	0.5779	0.5793	0.5807	0.5821	0.5835	0.5850	0.5864	2	5	7	9	12
36	0.5878	0.5892	0.5906	0.5920	0.5934	0.5948	0.5962	0.5976	0.5990	0.6004	2	5	7	9	12
37	0.6018	0.6032	0.6046	0.6060	0.6074	0.6088	0.6101	0.6115	0.6129	0.6143	2	5	7	9	12
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

sin x															
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences (Add)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
38	0.6157	0.6170	0.6184	0.6198	0.6211	0.6225	0.6239	0.6252	0.6266	0.6280	2	5	7	9	11
39	0.6293	0.6307	0.6320	0.6334	0.6347	0.6361	0.6374	0.6388	0.6401	0.6414	2	4	7	9	11
40	0.6428	0.6441	0.6455	0.6468	0.6481	0.6494	0.6508	0.6521	0.6534	0.6547	2	4	7	9	11
41	0.6561	0.6574	0.6587	0.6600	0.6613	0.6626	0.6639	0.6652	0.6665	0.6678	2	4	7	9	11
42	0.6691	0.6704	0.6717	0.6730	0.6743	0.6756	0.6769	0.6782	0.6794	0.6807	2	4	6	9	11
43	0.6820	0.6833	0.6845	0.6858	0.6871	0.6884	0.6896	0.6909	0.6921	0.6934	2	4	6	8	11
44	0.6947	0.6959	0.6972	0.6984	0.6997	0.7009	0.7022	0.7034	0.7046	0.7059	2	4	6	8	10
45	0.7071	0.7083	0.7096	0.7108	0.7120	0.7133	0.7145	0.7157	0.7169	0.7181	2	4	6	8	10
46	0.7193	0.7206	0.7218	0.7230	0.7242	0.7254	0.7266	0.7278	0.7290	0.7302	2	4	6	8	10
47	0.7314	0.7325	0.7337	0.7349	0.7361	0.7373	0.7385	0.7396	0.7408	0.7420	2	4	6	8	10
48	0.7431	0.7443	0.7455	0.7466	0.7478	0.7490	0.7501	0.7513	0.7524	0.7536	2	4	6	8	10
49	0.7547	0.7559	0.7570	0.7581	0.7593	0.7604	0.7615	0.7627	0.7638	0.7649	2	4	6	8	9
50	0.7660	0.7672	0.7683	0.7694	0.7705	0.7716	0.7727	0.7738	0.7749	0.7760	2	4	6	7	9
51	0.7771	0.7782	0.7793	0.7804	0.7815	0.7826	0.7837	0.7848	0.7859	0.7869	2	4	5	7	9
52	0.7880	0.7891	0.7902	0.7912	0.7923	0.7934	0.7944	0.7955	0.7965	0.7976	2	4	5	7	9
53	0.7986	0.7997	0.8007	0.8018	0.8028	0.8039	0.8049	0.8059	0.8070	0.8080	2	3	5	7	9
54	0.8090	0.8100	0.8111	0.8121	0.8131	0.8141	0.8151	0.8161	0.8171	0.8181	2	3	5	7	8
55	0.8192	0.8202	0.8211	0.8221	0.8231	0.8241	0.8251	0.8261	0.8271	0.8281	2	3	5	7	8
56	0.8290	0.8300	0.8310	0.8320	0.8329	0.8339	0.8348	0.8358	0.8368	0.8377	2	3	5	6	8
57	0.8387	0.8396	0.8406	0.8415	0.8425	0.8434	0.8443	0.8453	0.8462	0.8471	2	3	5	6	8
58	0.8480	0.8490	0.8499	0.8508	0.8517	0.8526	0.8536	0.8545	0.8554	0.8563	2	3	5	6	8
59	0.8572	0.8581	0.8590	0.8599	0.8607	0.8616	0.8625	0.8634	0.8643	0.8652	1	3	4	6	7
60	0.8660	0.8669	0.8678	0.8686	0.8695	0.8704	0.8712	0.8721	0.8729	0.8738	1	3	4	6	7
61	0.8746	0.8755	0.8763	0.8771	0.8780	0.8788	0.8796	0.8805	0.8813	0.8821	1	3	4	6	7
62	0.8829	0.8838	0.8846	0.8854	0.8862	0.8870	0.8878	0.8886	0.8894	0.8902	1	3	4	5	7
63	0.8910	0.8918	0.8926	0.8934	0.8942	0.8949	0.8957	0.8965	0.8973	0.8980	1	3	4	5	6
64	0.8988	0.8996	0.9003	0.9011	0.9018	0.9026	0.9033	0.9041	0.9048	0.9056	1	3	4	5	6
65	0.9063	0.9070	0.9078	0.9085	0.9092	0.9100	0.9107	0.9114	0.9121	0.9128	1	2	4	5	6
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

sin x

x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences (Add)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
66	0.9135	0.9143	0.9150	0.9157	0.9164	0.9171	0.9178	0.9184	0.9191	0.9198	1	2	3	5	6
67	0.9205	0.9212	0.9219	0.9225	0.9232	0.9239	0.9245	0.9252	0.9259	0.9265	1	2	3	4	6
68	0.9272	0.9278	0.9285	0.9291	0.9298	0.9304	0.9311	0.9317	0.9323	0.9330	1	2	3	4	5
69	0.9336	0.9342	0.9348	0.9354	0.9361	0.9367	0.9373	0.9379	0.9385	0.9391	1	2	3	4	5
70	0.9397	0.9403	0.9409	0.9415	0.9421	0.9426	0.9432	0.9438	0.9444	0.9449	1	2	3	4	5
71	0.9455	0.9461	0.9466	0.9472	0.9478	0.9483	0.9489	0.9494	0.9500	0.9505	1	2	3	4	5
72	0.9511	0.9516	0.9521	0.9527	0.9532	0.9537	0.9542	0.9548	0.9553	0.9558	1	2	3	3	4
73	0.9563	0.9568	0.9573	0.9578	0.9583	0.9588	0.9593	0.9598	0.9603	0.9608	1	2	2	3	4
74	0.9613	0.9617	0.9622	0.9627	0.9632	0.9636	0.9641	0.9646	0.9650	0.9655	1	2	2	3	4
75	0.9659	0.9664	0.9668	0.9673	0.9677	0.9681	0.9686	0.9690	0.9694	0.9699	1	1	2	3	4
76	0.9703	0.9707	0.9711	0.9715	0.9720	0.9724	0.9728	0.9732	0.9736	0.9740	1	1	2	3	3
77	0.9744	0.9748	0.9751	0.9755	0.9759	0.9763	0.9767	0.9770	0.9774	0.9778	1	1	2	3	3
78	0.9781	0.9785	0.9789	0.9792	0.9796	0.9799	0.9803	0.9806	0.9810	0.9813	1	1	2	2	3
79	0.9816	0.9820	0.9823	0.9826	0.9829	0.9833	0.9836	0.9839	0.9842	0.9845	1	1	2	2	3
80	0.9848	0.9851	0.9854	0.9857	0.9860	0.9863	0.9866	0.9869	0.9871	0.9874	0	1	1	2	2
81	0.9877	0.9880	0.9882	0.9885	0.9888	0.9890	0.9893	0.9895	0.9898	0.9900	0	1	1	2	2
82	0.9903	0.9905	0.9907	0.9910	0.9912	0.9914	0.9917	0.9919	0.9921	0.9923	0	1	1	2	2
83	0.9925	0.9928	0.9930	0.9932	0.9934	0.9936	0.9938	0.9940	0.9942	0.9943	0	1	1	1	2
84	0.9945	0.9947	0.9949	0.9951	0.9952	0.9954	0.9956	0.9957	0.9959	0.9960	0	1	1	1	2
85	0.9962	0.9963	0.9965	0.9966	0.9968	0.9969	0.9971	0.9972	0.9973	0.9974	0	0	1	1	1
86	0.9976	0.9977	0.9978	0.9979	0.9980	0.9981	0.9982	0.9983	0.9984	0.9985	0	0	1	1	1
87	0.9986	0.9987	0.9988	0.9989	0.9990	0.9990	0.9991	0.9992	0.9993	0.9993	0	0	0	1	1
88	0.9994	0.9995	0.9995	0.9996	0.9996	0.9997	0.9997	0.9997	0.9998	0.9998	0	0	0	0	0
89	0.9998	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

Cosine of an Angle

cos x															
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences (Subtract)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0
1	0.9998	0.9998	0.9998	0.9997	0.9997	0.9997	0.9996	0.9996	0.9995	0.9995	0	0	0	0	0
2	0.9994	0.9993	0.9993	0.9992	0.9991	0.9990	0.9990	0.9989	0.9988	0.9987	0	0	0	1	1
3	0.9986	0.9985	0.9984	0.9983	0.9982	0.9981	0.9980	0.9979	0.9978	0.9977	0	0	1	1	1
4	0.9976	0.9974	0.9973	0.9972	0.9971	0.9969	0.9968	0.9966	0.9965	0.9963	0	0	1	1	1
5	0.9962	0.9960	0.9959	0.9957	0.9956	0.9954	0.9952	0.9951	0.9949	0.9947	0	1	1	1	2
6	0.9945	0.9943	0.9942	0.9940	0.9938	0.9936	0.9934	0.9932	0.9930	0.9928	0	1	1	1	2
7	0.9925	0.9923	0.9921	0.9919	0.9917	0.9914	0.9912	0.9910	0.9907	0.9905	0	1	1	2	2
8	0.9903	0.9900	0.9898	0.9895	0.9893	0.9890	0.9888	0.9885	0.9882	0.9880	0	1	1	2	2
9	0.9877	0.9874	0.9871	0.9869	0.9866	0.9863	0.9860	0.9857	0.9854	0.9851	0	1	1	2	2
10	0.9848	0.9845	0.9842	0.9839	0.9836	0.9833	0.9829	0.9826	0.9823	0.9820	1	1	2	2	3
11	0.9816	0.9813	0.9810	0.9806	0.9803	0.9799	0.9796	0.9792	0.9789	0.9785	1	1	2	2	3
12	0.9781	0.9778	0.9774	0.9770	0.9767	0.9763	0.9759	0.9755	0.9751	0.9748	1	1	2	3	3
13	0.9744	0.9740	0.9736	0.9732	0.9728	0.9724	0.9720	0.9715	0.9711	0.9707	1	1	2	3	3
14	0.9703	0.9699	0.9694	0.9690	0.9686	0.9681	0.9677	0.9673	0.9668	0.9664	1	1	2	3	4
15	0.9659	0.9655	0.9650	0.9646	0.9641	0.9636	0.9632	0.9627	0.9622	0.9617	1	2	2	3	4
16	0.9613	0.9608	0.9603	0.9598	0.9593	0.9588	0.9583	0.9578	0.9573	0.9568	1	2	2	3	4
17	0.9563	0.9558	0.9553	0.9548	0.9542	0.9537	0.9532	0.9527	0.9521	0.9516	1	2	3	3	4
18	0.9511	0.9505	0.9500	0.9494	0.9489	0.9483	0.9478	0.9472	0.9466	0.9461	1	2	3	4	5
19	0.9455	0.9449	0.9444	0.9438	0.9432	0.9426	0.9421	0.9415	0.9409	0.9403	1	2	3	4	5
20	0.9397	0.9391	0.9385	0.9379	0.9373	0.9367	0.9361	0.9354	0.9348	0.9342	1	2	3	4	5
21	0.9336	0.9330	0.9323	0.9317	0.9311	0.9304	0.9298	0.9291	0.9285	0.9278	1	2	3	4	5
22	0.9272	0.9265	0.9259	0.9252	0.9245	0.9239	0.9232	0.9225	0.9219	0.9212	1	2	3	4	6
23	0.9205	0.9198	0.9191	0.9184	0.9178	0.9171	0.9164	0.9157	0.9150	0.9143	1	2	3	5	6
24	0.9135	0.9128	0.9121	0.9114	0.9107	0.9100	0.9092	0.9085	0.9078	0.9070	1	2	4	5	6
25	0.9063	0.9056	0.9048	0.9041	0.9033	0.9026	0.9018	0.9011	0.9003	0.8996	1	3	4	5	6
26	0.8988	0.8980	0.8973	0.8965	0.8957	0.8949	0.8942	0.8934	0.8926	0.8918	1	3	4	5	6
27	0.8910	0.8902	0.8894	0.8886	0.8878	0.8870	0.8862	0.8854	0.8846	0.8838	1	3	4	5	7
28	0.8829	0.8821	0.8813	0.8805	0.8796	0.8788	0.8780	0.8771	0.8763	0.8755	1	3	4	6	7
29	0.8746	0.8738	0.8729	0.8721	0.8712	0.8704	0.8695	0.8686	0.8678	0.8669	1	3	4	6	7
30	0.8660	0.8652	0.8643	0.8634	0.8625	0.8616	0.8607	0.8599	0.8590	0.8581	1	3	4	6	7
31	0.8572	0.8563	0.8554	0.8545	0.8536	0.8526	0.8517	0.8508	0.8499	0.8490	2	3	5	6	8
32	0.8480	0.8471	0.8462	0.8453	0.8443	0.8434	0.8425	0.8415	0.8406	0.8396	2	3	5	6	8
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

cos x

x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences (Subtract)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
33	0.8387	0.8377	0.8368	0.8358	0.8348	0.8339	0.8329	0.8320	0.8310	0.8300	2	3	5	6	8
34	0.8290	0.8281	0.8271	0.8261	0.8251	0.8241	0.8231	0.8221	0.8211	0.8202	2	3	5	7	8
35	0.8192	0.8181	0.8171	0.8161	0.8151	0.8141	0.8131	0.8121	0.8111	0.8100	2	3	5	7	8
36	0.8090	0.8080	0.8070	0.8059	0.8049	0.8039	0.8028	0.8018	0.8007	0.7997	2	3	5	7	9
37	0.7986	0.7976	0.7965	0.7955	0.7944	0.7934	0.7923	0.7912	0.7902	0.7891	2	4	5	7	9
38	0.7880	0.7869	0.7859	0.7848	0.7837	0.7826	0.7815	0.7804	0.7793	0.7782	2	4	5	7	9
39	0.7771	0.7760	0.7749	0.7738	0.7727	0.7716	0.7705	0.7694	0.7683	0.7672	2	4	6	7	9
40	0.7660	0.7649	0.7638	0.7627	0.7615	0.7604	0.7593	0.7581	0.7570	0.7559	2	4	6	8	9
41	0.7547	0.7536	0.7524	0.7513	0.7501	0.7490	0.7478	0.7466	0.7455	0.7443	2	4	6	8	10
42	0.7431	0.7420	0.7408	0.7396	0.7385	0.7373	0.7361	0.7349	0.7337	0.7325	2	4	6	8	10
43	0.7314	0.7302	0.7290	0.7278	0.7266	0.7254	0.7242	0.7230	0.7218	0.7206	2	4	6	8	10
44	0.7193	0.7181	0.7169	0.7157	0.7145	0.7133	0.7120	0.7108	0.7096	0.7083	2	4	6	8	10
45	0.7071	0.7059	0.7046	0.7034	0.7022	0.7009	0.6997	0.6984	0.6972	0.6959	2	4	6	8	10
46	0.6947	0.6934	0.6921	0.6909	0.6896	0.6884	0.6871	0.6858	0.6845	0.6833	2	4	6	9	11
47	0.6820	0.6807	0.6794	0.6782	0.6769	0.6756	0.6743	0.6730	0.6717	0.6704	2	4	6	9	11
48	0.6691	0.6678	0.6665	0.6652	0.6639	0.6626	0.6613	0.6600	0.6587	0.6574	2	4	7	9	11
49	0.6561	0.6547	0.6534	0.6521	0.6508	0.6494	0.6481	0.6468	0.6455	0.6441	2	4	7	9	11
50	0.6428	0.6414	0.6401	0.6388	0.6374	0.6361	0.6347	0.6334	0.6320	0.6307	2	4	7	9	11
51	0.6293	0.6280	0.6266	0.6252	0.6239	0.6225	0.6211	0.6198	0.6184	0.6170	2	5	7	9	11
52	0.6157	0.6143	0.6129	0.6115	0.6101	0.6088	0.6074	0.6060	0.6046	0.6032	2	5	7	9	12
53	0.6018	0.6004	0.5990	0.5976	0.5962	0.5948	0.5934	0.5920	0.5906	0.5892	2	5	7	9	12
54	0.5878	0.5864	0.5850	0.5835	0.5821	0.5807	0.5793	0.5779	0.5764	0.5750	2	5	7	9	12
55	0.5736	0.5721	0.5707	0.5693	0.5678	0.5664	0.5650	0.5635	0.5621	0.5606	2	5	7	10	12
56	0.5592	0.5577	0.5563	0.5548	0.5534	0.5519	0.5505	0.5490	0.5476	0.5461	2	5	7	10	12
57	0.5446	0.5432	0.5417	0.5402	0.5388	0.5373	0.5358	0.5344	0.5329	0.5314	2	5	7	10	12
58	0.5299	0.5284	0.5270	0.5255	0.5240	0.5225	0.5210	0.5195	0.5180	0.5165	2	5	7	10	12
59	0.5150	0.5135	0.5120	0.5105	0.5090	0.5075	0.5060	0.5045	0.5030	0.5015	3	5	8	10	13
60	0.5000	0.4985	0.4970	0.4955	0.4939	0.4924	0.4909	0.4894	0.4879	0.4863	3	5	8	10	13
61	0.4848	0.4833	0.4818	0.4802	0.4787	0.4772	0.4756	0.4741	0.4726	0.4710	3	5	8	10	13
62	0.4695	0.4679	0.4664	0.4648	0.4633	0.4617	0.4602	0.4586	0.4571	0.4555	3	5	8	10	13
63	0.4540	0.4524	0.4509	0.4493	0.4478	0.4462	0.4446	0.4431	0.4415	0.4399	3	5	8	10	13
64	0.4384	0.4368	0.4352	0.4337	0.4321	0.4305	0.4289	0.4274	0.4258	0.4242	3	5	8	11	13
65	0.4226	0.4210	0.4195	0.4179	0.4163	0.4147	0.4131	0.4115	0.4099	0.4083	3	5	8	11	13
66	0.4067	0.4051	0.4035	0.4019	0.4003	0.3987	0.3971	0.3955	0.3939	0.3923	3	5	8	11	14
67	0.3907	0.3891	0.3875	0.3859	0.3843	0.3827	0.3811	0.3795	0.3778	0.3762	3	5	8	11	14
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

cos x															
x	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	Mean Differences (Subtract)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
68	0.3746	0.3730	0.3714	0.3697	0.3681	0.3665	0.3649	0.3633	0.3616	0.3600	3	5	8	11	14
69	0.3584	0.3567	0.3551	0.3535	0.3518	0.3502	0.3486	0.3469	0.3453	0.3437	3	5	8	11	14
70	0.3420	0.3404	0.3387	0.3371	0.3355	0.3338	0.3322	0.3305	0.3289	0.3272	3	5	8	11	14
71	0.3256	0.3239	0.3223	0.3206	0.3190	0.3173	0.3156	0.3140	0.3123	0.3107	3	6	8	11	14
72	0.3090	0.3074	0.3057	0.3040	0.3024	0.3007	0.2990	0.2974	0.2957	0.2940	3	6	8	11	14
73	0.2924	0.2907	0.2890	0.2874	0.2857	0.2840	0.2823	0.2807	0.2790	0.2773	3	6	8	11	14
74	0.2756	0.2740	0.2723	0.2706	0.2689	0.2672	0.2656	0.2639	0.2622	0.2605	3	6	8	11	14
75	0.2588	0.2571	0.2554	0.2538	0.2521	0.2504	0.2487	0.2470	0.2453	0.2436	3	6	8	11	14
76	0.2419	0.2402	0.2385	0.2368	0.2351	0.2334	0.2317	0.2300	0.2284	0.2267	3	6	8	11	14
77	0.2250	0.2233	0.2215	0.2198	0.2181	0.2164	0.2147	0.2130	0.2113	0.2096	3	6	9	11	14
78	0.2079	0.2062	0.2045	0.2028	0.2011	0.1994	0.1977	0.1959	0.1942	0.1925	3	6	9	11	14
79	0.1908	0.1891	0.1874	0.1857	0.1840	0.1822	0.1805	0.1788	0.1771	0.1754	3	6	9	11	14
80	0.1736	0.1719	0.1702	0.1685	0.1668	0.1650	0.1633	0.1616	0.1599	0.1582	3	6	9	12	14
81	0.1564	0.1547	0.1530	0.1513	0.1495	0.1478	0.1461	0.1444	0.1426	0.1409	3	6	9	12	14
82	0.1392	0.1374	0.1357	0.1340	0.1323	0.1305	0.1288	0.1271	0.1253	0.1236	3	6	9	12	14
83	0.1219	0.1201	0.1184	0.1167	0.1149	0.1132	0.1115	0.1097	0.1080	0.1063	3	6	9	12	14
84	0.1045	0.1028	0.1011	0.0993	0.0976	0.0958	0.0941	0.0924	0.0906	0.0889	3	6	9	12	14
85	0.0872	0.0854	0.0837	0.0819	0.0802	0.0785	0.0767	0.0750	0.0732	0.0715	3	6	9	12	14
86	0.0698	0.0680	0.0663	0.0645	0.0628	0.0610	0.0593	0.0576	0.0558	0.0541	3	6	9	12	15
87	0.0523	0.0506	0.0488	0.0471	0.0454	0.0436	0.0419	0.0401	0.0384	0.0366	3	6	9	12	15
88	0.0349	0.0332	0.0314	0.0297	0.0279	0.0262	0.0244	0.0227	0.0209	0.0192	3	6	9	12	15
89	0.0175	0.0157	0.0140	0.0122	0.0105	0.0087	0.0070	0.0052	0.0035	0.0017	3	6	9	12	15
x	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1	2	3	4	5
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

Tangent of an Angle

x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences (Add)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157	3	6	9	12	15
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332	3	6	9	12	15
2	0.0349	0.0367	0.0384	0.0402	0.0419	0.0437	0.0454	0.0472	0.0489	0.0507	3	6	9	12	15
3	0.0524	0.0542	0.0559	0.0577	0.0594	0.0612	0.0629	0.0647	0.0664	0.0682	3	6	9	12	15
4	0.0699	0.0717	0.0734	0.0752	0.0769	0.0787	0.0805	0.0822	0.0840	0.0857	3	6	9	12	15
5	0.0875	0.0892	0.0910	0.0928	0.0945	0.0963	0.0981	0.0998	0.1016	0.1033	3	6	9	12	15
6	0.1051	0.1069	0.1086	0.1104	0.1122	0.1139	0.1157	0.1175	0.1192	0.1210	3	6	9	12	15
7	0.1228	0.1246	0.1263	0.1281	0.1299	0.1317	0.1334	0.1352	0.1370	0.1388	3	6	9	12	15
8	0.1405	0.1423	0.1441	0.1459	0.1477	0.1495	0.1512	0.1530	0.1548	0.1566	3	6	9	12	15
9	0.1584	0.1602	0.1620	0.1638	0.1655	0.1673	0.1691	0.1709	0.1727	0.1745	3	6	9	12	15
10	0.1763	0.1781	0.1799	0.1817	0.1835	0.1853	0.1871	0.1890	0.1908	0.1926	3	6	9	12	15
11	0.1944	0.1962	0.1980	0.1998	0.2016	0.2035	0.2053	0.2071	0.2089	0.2107	3	6	9	12	15
12	0.2126	0.2144	0.2162	0.2180	0.2199	0.2217	0.2235	0.2254	0.2272	0.2290	3	6	9	12	15
13	0.2309	0.2327	0.2345	0.2364	0.2382	0.2401	0.2419	0.2438	0.2456	0.2475	3	6	9	12	15
14	0.2493	0.2512	0.2530	0.2549	0.2568	0.2586	0.2605	0.2623	0.2642	0.2661	3	6	9	12	16
15	0.2679	0.2698	0.2717	0.2736	0.2754	0.2773	0.2792	0.2811	0.2830	0.2849	3	6	9	13	16
16	0.2867	0.2886	0.2905	0.2924	0.2943	0.2962	0.2981	0.3000	0.3019	0.3038	3	6	9	13	16
17	0.3057	0.3076	0.3096	0.3115	0.3134	0.3153	0.3172	0.3191	0.3211	0.3230	3	6	10	13	16
18	0.3249	0.3269	0.3288	0.3307	0.3327	0.3346	0.3365	0.3385	0.3404	0.3424	3	6	10	13	16
19	0.3443	0.3463	0.3482	0.3502	0.3522	0.3541	0.3561	0.3581	0.3600	0.3620	3	7	10	13	16
20	0.3640	0.3659	0.3679	0.3699	0.3719	0.3739	0.3759	0.3779	0.3799	0.3819	3	7	10	13	17
21	0.3839	0.3859	0.3879	0.3899	0.3919	0.3939	0.3959	0.3979	0.4000	0.4020	3	7	10	13	17
22	0.4040	0.4061	0.4081	0.4101	0.4122	0.4142	0.4163	0.4183	0.4204	0.4224	3	7	10	14	17
23	0.4245	0.4265	0.4286	0.4307	0.4327	0.4348	0.4369	0.4390	0.4411	0.4431	3	7	10	14	17
24	0.4452	0.4473	0.4494	0.4515	0.4536	0.4557	0.4578	0.4599	0.4621	0.4642	4	7	11	14	18
25	0.4663	0.4684	0.4706	0.4727	0.4748	0.4770	0.4791	0.4813	0.4834	0.4856	4	7	11	14	18
26	0.4877	0.4899	0.4921	0.4942	0.4964	0.4986	0.5008	0.5029	0.5051	0.5073	4	7	11	15	18
27	0.5095	0.5117	0.5139	0.5161	0.5184	0.5206	0.5228	0.5250	0.5272	0.5295	4	7	11	15	18
28	0.5317	0.5340	0.5362	0.5384	0.5407	0.5430	0.5452	0.5475	0.5498	0.5520	4	8	11	15	19
29	0.5543	0.5566	0.5589	0.5612	0.5635	0.5658	0.5681	0.5704	0.5727	0.5750	4	8	12	15	19
30	0.5774	0.5797	0.5820	0.5844	0.5867	0.5890	0.5914	0.5938	0.5961	0.5985	4	8	12	16	20
31	0.6009	0.6032	0.6056	0.6080	0.6104	0.6128	0.6152	0.6176	0.6200	0.6224	4	8	12	16	20
32	0.6249	0.6273	0.6297	0.6322	0.6346	0.6371	0.6395	0.6420	0.6445	0.6469	4	8	12	16	20
33	0.6494	0.6519	0.6544	0.6569	0.6594	0.6619	0.6644	0.6669	0.6694	0.6720	4	8	13	17	21
34	0.6745	0.6771	0.6796	0.6822	0.6847	0.6873	0.6899	0.6924	0.6950	0.6976	4	9	13	17	21
35	0.7002	0.7028	0.7054	0.7080	0.7107	0.7133	0.7159	0.7186	0.7212	0.7239	4	9	13	18	22
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

tan x

x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences (Add)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
36	0.7265	0.7292	0.7319	0.7346	0.7373	0.7400	0.7427	0.7454	0.7481	0.7508	4	9	13	18	23
37	0.7536	0.7563	0.7590	0.7618	0.7646	0.7673	0.7701	0.7729	0.7757	0.7785	5	9	14	18	23
38	0.7813	0.7841	0.7869	0.7898	0.7926	0.7954	0.7983	0.8012	0.8040	0.8069	5	9	14	19	24
39	0.8098	0.8127	0.8156	0.8185	0.8214	0.8243	0.8273	0.8302	0.8332	0.8361	5	10	15	20	24
40	0.8391	0.8421	0.8451	0.8481	0.8511	0.8541	0.8571	0.8601	0.8632	0.8662	5	10	15	20	25
41	0.8693	0.8724	0.8754	0.8785	0.8816	0.8847	0.8878	0.8910	0.8941	0.8972	5	10	16	21	26
42	0.9004	0.9036	0.9067	0.9099	0.9131	0.9163	0.9195	0.9228	0.9260	0.9293	5	11	16	21	27
43	0.9325	0.9358	0.9391	0.9424	0.9457	0.9490	0.9523	0.9556	0.9590	0.9623	6	11	17	22	28
44	0.9657	0.9691	0.9725	0.9759	0.9793	0.9827	0.9861	0.9896	0.9930	0.9965	6	11	17	23	29
45	1.0000	1.0035	1.0070	1.0105	1.0141	1.0176	1.0212	1.0247	1.0283	1.0319	6	12	18	24	30
46	1.0355	1.0392	1.0428	1.0464	1.0501	1.0538	1.0575	1.0612	1.0649	1.0686	6	12	18	25	31
47	1.0724	1.0761	1.0799	1.0837	1.0875	1.0913	1.0951	1.0990	1.1028	1.1067	6	13	19	25	32
48	1.1106	1.1145	1.1184	1.1224	1.1263	1.1303	1.1343	1.1383	1.1423	1.1463	7	13	20	26	33
49	1.1504	1.1544	1.1585	1.1626	1.1667	1.1708	1.1750	1.1792	1.1833	1.1875	7	14	21	28	34
50	1.1918	1.1960	1.2002	1.2045	1.2088	1.2131	1.2174	1.2218	1.2261	1.2305	7	14	22	29	36
51	1.2349	1.2393	1.2437	1.2482	1.2527	1.2572	1.2617	1.2662	1.2708	1.2753	7	15	22	30	38
52	1.2799	1.2846	1.2892	1.2938	1.2985	1.3032	1.3079	1.3127	1.3175	1.3222	8	16	24	31	39
53	1.3270	1.3319	1.3367	1.3416	1.3465	1.3514	1.3564	1.3613	1.3663	1.3713	8	16	25	33	41
54	1.3764	1.3814	1.3865	1.3916	1.3968	1.4019	1.4071	1.4124	1.4176	1.4229	9	17	26	34	43
55	1.4281	1.4335	1.4388	1.4442	1.4496	1.4550	1.4605	1.4659	1.4715	1.4770	9	18	27	36	45
56	1.4826	1.4882	1.4938	1.4994	1.5051	1.5108	1.5166	1.5224	1.5282	1.5340	10	19	29	38	48
57	1.5399	1.5458	1.5517	1.5577	1.5637	1.5697	1.5757	1.5818	1.5880	1.5941	10	20	30	40	50
58	1.6003	1.6066	1.6128	1.6191	1.6255	1.6319	1.6383	1.6447	1.6512	1.6577	11	21	32	43	53
59	1.6643	1.6709	1.6775	1.6842	1.6909	1.6977	1.7045	1.7113	1.7182	1.7251	11	23	34	45	56
60	1.7321	1.7391	1.7461	1.7532	1.7603	1.7675	1.7747	1.7820	1.7893	1.7966	12	24	36	48	60
61	1.8040	1.8115	1.8190	1.8265	1.8341	1.8418	1.8495	1.8572	1.8650	1.8728	13	26	38	51	64
62	1.8807	1.8887	1.8967	1.9047	1.9128	1.9210	1.9292	1.9375	1.9458	1.9542	14	27	41	55	68
63	1.9626	1.9711	1.9797	1.9883	1.9970	2.0057	2.0145	2.0233	2.0323	2.0413	15	29	44	58	73
64	2.0503	2.0594	2.0686	2.0778	2.0872	2.0965	2.1060	2.1155	2.1251	2.1348	16	31	47	63	78
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences (Add)				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1′	2′	3′	4′	5′
65	2.1445	2.1543	2.1642	2.1742	2.1842	2.1943	2.2045	2.2148	2.2251	2.2355	17	34	51	68	85
66	2.2460	2.2566	2.2673	2.2781	2.2889	2.2998	2.3109	2.3220	2.3332	2.3445	18	37	55	73	91
67	2.3559	2.3673	2.3789	2.3906	2.4023	2.4142	2.4262	2.4383	2.4504	2.4627	20	40	59	79	99
68	2.4751	2.4876	2.5002	2.5129	2.5257	2.5386	2.5517	2.5649	2.5782	2.5916	22	43	65	87	108
69	2.6051	2.6187	2.6325	2.6464	2.6605	2.6746	2.6889	2.7034	2.7179	2.7326	24	47	71	95	119
70	2.7475	2.7625	2.7776	2.7929	2.8083	2.8239	2.8397	2.8556	2.8716	2.8878	26	52	78	104	131
71	2.9042	2.9208	2.9375	2.9544	2.9714	2.9887	3.0061	3.0237	3.0415	3.0595	29	58	87	115	144
72	3.0777	3.0961	3.1146	3.1334	3.1524	3.1716	3.1910	3.2106	3.2305	3.2506	32	64	96	129	161
73	3.2709	3.2914	3.3122	3.3332	3.3544	3.3759	3.3977	3.4197	3.4420	3.4646	36	72	108	144	180
74	3.4874	3.5105	3.5339	3.5576	3.5816	3.6059	3.6305	3.6554	3.6806	3.7062	41	81	122	163	204
75	3.7321	3.7583	3.7848	3.8118	3.8391	3.8667	3.8947	3.9232	3.9520	3.9812	46	92	139	185	232
76	4.0108	4.0408	4.0713	4.1022	4.1335	4.1653	4.1976	4.2303	4.2635	4.2972	Mean Differences ceases to be sufficiently accurate				
77	4.3315	4.3662	4.4015	4.4373	4.4737	4.5107	4.5483	4.5864	4.6252	4.6646					
78	4.7046	4.7453	4.7867	4.8288	4.8716	4.9152	4.9594	5.0045	5.0504	5.0970					
79	5.1446	5.1929	5.2422	5.2924	5.3435	5.3955	5.4486	5.5026	5.5578	5.6140					
80	5.6713	5.7297	5.7894	5.8502	5.9124	5.9758	6.0405	6.1066	6.1742	6.2432					
81	6.3138	6.3859	6.4596	6.5350	6.6122	6.6912	6.7720	6.8548	6.9395	7.0264					
82	7.1154	7.2066	7.3002	7.3962	7.4947	7.5958	7.6996	7.8062	7.9158	8.0285					
83	8.1443	8.2636	8.3863	8.5126	8.6427	8.7769	8.9152	9.0579	9.2052	9.3572					
84	9.5144	9.6768	9.8448	10.019	10.199	10.385	10.579	10.780	10.988	11.205					
85	11.430	11.664	11.909	12.163	12.429	12.706	12.996	13.300	13.617	13.951					
86	14.301	14.669	15.056	15.464	15.895	16.350	16.832	17.343	17.886	18.464					
87	19.081	19.740	20.446	21.205	22.022	22.904	23.859	24.898	26.031	27.271					
88	28.636	30.145	31.821	33.694	35.801	38.188	40.917	44.066	47.740	52.081					
89	57.290	63.657	71.615	81.847	95.489	114.59	143.24	190.98	286.48	572.96					
x	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°					

Table title: tan x

Square Root

	$\sqrt{x}$										Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044	0	1	1	2	2	3	3	4	4
1.1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091	0	1	1	2	2	3	3	4	4
1.2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136	0	1	1	2	2	3	3	4	4
1.3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179	0	1	1	2	2	3	3	3	4
1.4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221	0	1	1	2	2	2	3	3	4
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261	0	1	1	2	2	2	3	3	4
1.6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300	0	1	1	2	2	2	3	3	3
1.7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338	0	1	1	2	2	2	3	3	3
1.8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375	0	1	1	1	2	2	3	3	3
1.9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411	0	1	1	1	2	2	3	3	3
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446	0	1	1	1	2	2	2	3	3
2.1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480	0	1	1	1	2	2	2	3	3
2.2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513	0	1	1	1	2	2	2	3	3
2.3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546	0	1	1	1	2	2	2	3	3
2.4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578	0	1	1	1	2	2	2	3	3
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609	0	1	1	1	2	2	2	3	3
2.6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640	0	1	1	1	2	2	2	2	3
2.7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670	0	1	1	1	2	2	2	2	3
2.8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700	0	1	1	1	1	2	2	2	3
2.9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729	0	1	1	1	1	2	2	2	3
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758	0	1	1	1	1	2	2	2	3
3.1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786	0	1	1	1	1	2	2	2	3
3.2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814	0	1	1	1	1	2	2	2	2
3.3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841	0	1	1	1	1	2	2	2	2
3.4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868	0	1	1	1	1	2	2	2	2
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895	0	1	1	1	1	2	2	2	2
3.6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921	0	1	1	1	1	2	2	2	2
3.7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947	0	1	1	1	1	2	2	2	2
3.8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972	0	1	1	1	1	2	2	2	2
3.9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997	0	1	1	1	1	2	2	2	2
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022	0	0	1	1	1	1	2	2	2
4.1	2.025	2.027	2.030	2.032	2.035	2.037	2.040	2.042	2.045	2.047	0	0	1	1	1	1	2	2	2
4.2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071	0	0	1	1	1	1	2	2	2
4.3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095	0	0	1	1	1	1	2	2	2
4.4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119	0	0	1	1	1	1	2	2	2
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2.142	0	0	1	1	1	1	2	2	2
4.6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166	0	0	1	1	1	1	2	2	2
4.7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189	0	0	1	1	1	1	2	2	2
4.8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211	0	0	1	1	1	1	2	2	2
4.9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234	0	0	1	1	1	1	2	2	2
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256	0	0	1	1	1	1	2	2	2
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

$\sqrt{x}$											Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278	0	0	1	1	1	1	2	2	2
5.2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300	0	0	1	1	1	1	2	2	2
5.3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322	0	0	1	1	1	1	2	2	2
5.4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343	0	0	1	1	1	1	1	2	2
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364	0	0	1	1	1	1	1	2	2
5.6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385	0	0	1	1	1	1	1	2	2
5.7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406	0	0	1	1	1	1	1	2	2
5.8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427	0	0	1	1	1	1	1	2	2
5.9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447	0	0	1	1	1	1	1	2	2
6.0	2.449	2.452	2.454	2.456	2.458	2.460	2.462	2.464	2.466	2.468	0	0	1	1	1	1	1	2	2
6.1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488	0	0	1	1	1	1	1	2	2
6.2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508	0	0	1	1	1	1	1	2	2
6.3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528	0	0	1	1	1	1	1	2	2
6.4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548	0	0	1	1	1	1	1	2	2
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567	0	0	1	1	1	1	1	2	2
6.6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587	0	0	1	1	1	1	1	2	2
6.7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606	0	0	1	1	1	1	1	2	2
6.8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625	0	0	1	1	1	1	1	2	2
6.9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644	0	0	1	1	1	1	1	2	2
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663	0	0	1	1	1	1	1	2	2
7.1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681	0	0	1	1	1	1	1	1	2
7.2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700	0	0	1	1	1	1	1	1	2
7.3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718	0	0	1	1	1	1	1	1	2
7.4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737	0	0	1	1	1	1	1	1	2
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755	0	0	1	1	1	1	1	1	2
7.6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773	0	0	1	1	1	1	1	1	2
7.7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791	0	0	1	1	1	1	1	1	2
7.8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809	0	0	1	1	1	1	1	1	2
7.9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827	0	0	1	1	1	1	1	1	2
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844	0	0	1	1	1	1	1	1	2
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Math F2 – (DOOKS (DUMMY) Sept 4.indb 439

$\sqrt{x}$											Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
8.1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862	0	0	1	1	1	1	1	1	2
8.2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.877	2.879	0	0	1	1	1	1	1	1	2
8.3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897	0	0	1	1	1	1	1	1	2
8.4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914	0	0	1	1	1	1	1	1	2
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931	0	0	1	1	1	1	1	1	2
8.6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948	0	0	1	1	1	1	1	1	2
8.7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965	0	0	1	1	1	1	1	1	2
8.8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982	0	0	1	1	1	1	1	1	2
8.9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998	0	0	1	1	1	1	1	1	2
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015	0	0	0	1	1	1	1	1	1
9.1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032	0	0	0	1	1	1	1	1	1
9.2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048	0	0	0	1	1	1	1	1	1
9.3	3.050	3.051	3.053	3.055	3.056	3.058	3.059	3.061	3.063	3.064	0	0	0	1	1	1	1	1	1
9.4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081	0	0	0	1	1	1	1	1	1
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097	0	0	0	1	1	1	1	1	1
9.6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113	0	0	0	1	1	1	1	1	1
9.7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129	0	0	0	1	1	1	1	1	1
9.8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145	0	0	0	1	1	1	1	1	1
9.9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161	0	0	0	1	1	1	1	1	1
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Square Root

	$\sqrt{x}$										Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302	2	3	5	6	8	9	11	12	14
11	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450	1	3	4	6	7	9	10	12	13
12	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592	1	3	4	6	7	8	10	11	13
13	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728	1	3	4	5	7	8	10	11	12
14	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860	1	3	4	5	7	8	9	11	12
15	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987	1	3	4	5	6	8	9	10	11
16	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111	1	2	4	5	6	7	9	10	11
17	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231	1	2	4	5	6	7	8	10	11
18	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347	1	2	3	5	6	7	8	9	10
19	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461	1	2	3	5	6	7	8	9	10
20	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572	1	2	3	4	6	7	8	9	10
21	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680	1	2	3	4	5	6	8	9	10
22	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785	1	2	3	4	5	6	7	8	9
23	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889	1	2	3	4	5	6	7	8	9
24	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990	1	2	3	4	5	6	7	8	9
25	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089	1	2	3	4	5	6	7	8	9
26	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187	1	2	3	4	5	6	7	8	9
27	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282	1	2	3	4	5	6	7	8	9
28	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376	1	2	3	4	5	6	7	7	8
29	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468	1	2	3	4	5	6	6	7	8
30	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559	1	2	3	4	5	5	6	7	8
31	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648	1	2	3	4	4	5	6	7	8
32	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736	1	2	3	4	4	5	6	7	8
33	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822	1	2	3	3	4	5	6	7	8
34	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908	1	2	3	3	4	5	6	7	8
35	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992	1	2	3	3	4	5	6	7	8
36	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075	1	2	2	3	4	5	6	7	7
37	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156	1	2	2	3	4	5	6	7	7
38	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237	1	2	2	3	4	5	6	6	7
39	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317	1	2	2	3	4	5	6	6	7
40	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395	1	2	2	3	4	5	6	6	7
41	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473	1	2	2	3	4	5	5	6	7
42	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550	1	2	2	3	4	5	5	6	7
43	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626	1	2	2	3	4	5	5	6	7
44	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701	1	1	2	3	4	4	5	6	7
45	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775	1	1	2	3	4	4	5	6	7
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

$\sqrt{x}$											Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
46	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848	1	1	2	3	4	4	5	6	7
47	6.856	6.863	6.870	6.877	6.885	6.892	6.899	6.907	6.914	6.921	1	1	2	3	4	4	5	6	7
48	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993	1	1	2	3	4	4	5	6	6
49	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064	1	1	2	3	4	4	5	6	6
50	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134	1	1	2	3	4	4	5	6	6
51	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204	1	1	2	3	3	4	5	6	6
52	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273	1	1	2	3	3	4	5	6	6
53	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342	1	1	2	3	3	4	5	5	6
54	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409	1	1	2	3	3	4	5	5	6
55	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477	1	1	2	3	3	4	5	5	6
56	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543	1	1	2	3	3	4	5	5	6
57	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609	1	1	2	3	3	4	5	5	6
58	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675	1	1	2	3	3	4	5	5	6
59	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740	1	1	2	3	3	4	5	5	6
60	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804	1	1	2	3	3	4	5	5	6
61	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868	1	1	2	3	3	4	4	5	6
62	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931	1	1	2	3	3	4	4	5	6
63	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994	1	1	2	3	3	4	4	5	6
64	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056	1	1	2	2	3	4	4	5	6
65	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118	1	1	2	2	3	4	4	5	6
66	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179	1	1	2	2	3	4	4	5	6
67	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240	1	1	2	2	3	4	4	5	5
68	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301	1	1	2	2	3	4	4	5	5
69	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361	1	1	2	2	3	4	4	5	5
70	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420	1	1	2	2	3	4	4	5	5
71	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479	1	1	2	2	3	4	4	5	5
72	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538	1	1	2	2	3	4	4	5	5
73	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597	1	1	2	2	3	3	4	5	5
74	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654	1	1	2	2	3	3	4	5	5
75	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712	1	1	2	2	3	3	4	5	5
76	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769	1	1	2	2	3	3	4	5	5
77	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826	1	1	2	2	3	3	4	5	5
78	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883	1	1	2	2	3	3	4	5	5
79	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939	1	1	2	2	3	3	4	4	5
80	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994	1	1	2	2	3	3	4	4	5
81	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050	1	1	2	2	3	3	4	4	5
82	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105	1	1	2	2	3	3	4	4	5
83	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160	1	1	2	2	3	3	4	4	5
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

	$\sqrt{x}$										Mean Differences (Add)								
x	0'	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
84	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214	1	1	2	2	3	3	4	4	5
85	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268	1	1	2	2	3	3	4	4	3
86	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322	1	1	2	2	3	3	4	4	5
87	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375	1	1	2	2	3	3	4	4	5
88	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429	1	1	2	2	3	3	4	4	5
89	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482	1	1	2	2	3	3	4	4	5
90	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534	1	1	2	2	3	3	4	4	5
91	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586	1	1	2	2	3	3	4	4	5
93	9.644	9.649	9.654	9.659	9.664	9.670	9,675	9.680	9.685	9.690	1	1	2	2	3	3	4	4	5
94	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742	1	1	2	2	3	3	4	4	5
95	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793	1	1	2	2	3	3	4	4	5
96	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844	1	1	2	2	3	3	4	4	5
97	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894	1	1	2	2	3	3	4	4	5
98	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945	1	1	2	2	3	3	4	4	5
99	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995	1	1	2	2	3	3	4	4	5
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Reciprocal

$\frac{1}{x}$ — Mean Differences (Subtract)

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.0000	0.9901	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346	0.9259	0.9174	9	18	27	37	46	55	64	73	82
1.1	0.9091	0.9009	0.8929	0.8850	0.8772	0.8696	0.8621	0.8547	0.8475	0.8403	8	15	23	30	38	46	53	61	68
1.2	0.8333	0.8264	0.8197	0.8130	0.8065	0.8000	0.7937	0.7874	0.7813	0.7752	6	13	19	26	32	39	45	51	58
1.3	0.7692	0.7634	0.7576	0.7519	0.7463	0.7407	0.7353	0.7299	0.7246	0.7194	6	11	17	22	28	33	39	44	49
1.4	0.7143	0.7092	0.7042	0.6993	0.6944	0.6897	0.6849	0.6803	0.6757	0.6711	5	10	14	19	24	29	33	38	43
1.5	0.6667	0.6623	0.6579	0.6536	0.6494	0.6452	0.6410	0.6369	0.6329	0.6289	4	8	13	17	21	25	29	33	38
1.6	0.6250	0.6211	0.6173	0.6135	0.6098	0.6061	0.6024	0.5988	0.5952	0.5917	4	7	11	15	18	22	26	29	33
1.7	0.5882	0.5848	0.5814	0.5780	0.5747	0.5714	0.5682	0.5650	0.5618	0.5587	3	7	10	13	16	20	23	26	29
1.8	0.5556	0.5525	0.5495	0.5464	0.5435	0.5405	0.5376	0.5348	0.5319	0.5291	3	6	9	12	15	18	21	23	26
1.9	0.5263	0.5236	0.5208	0.5181	0.5155	0.5128	0.5102	0.5076	0.5051	0.5025	3	5	8	11	13	16	18	21	24
2.0	0.5000	0.4975	0.4950	0.4926	0.4902	0.4878	0.4854	0.4831	0.4808	0.4785	2	5	7	10	12	14	17	19	21
2.1	0.4762	0.4739	0.4717	0.4695	0.4673	0.4651	0.4630	0.4608	0.4587	0.4566	2	4	7	9	11	13	15	17	19
2.2	0.4545	0.4525	0.4505	0.4484	0.4464	0.4444	0.4425	0.4405	0.4386	0.4367	2	4	6	8	10	12	14	16	18
2.3	0.4348	0.4329	0.4310	0.4292	0.4274	0.4255	0.4237	0.4219	0.4202	0.4184	2	4	5	7	9	11	13	15	16
2.4	0.4167	0.4149	0.4132	0.4115	0.4098	0.4082	0.4065	0.4049	0.4032	0.4016	2	3	5	7	8	10	12	13	15
2.5	0.4000	0.3984	0.3968	0.3953	0.3937	0.3922	0.3906	0.3891	0.3876	0.3861	2	3	5	6	8	9	11	12	14
2.6	0.3846	0.3831	0.3817	0.3802	0.3788	0.3774	0.3759	0.3745	0.3731	0.3717	1	3	4	6	7	9	10	11	13
2.7	0.3704	0.3690	0.3676	0.3663	0.3650	0.3636	0.3623	0.3610	0.3597	0.3584	1	3	4	5	7	8	9	11	12
2.8	0.3571	0.3559	0.3546	0.3534	0.3521	0.3509	0.3497	0.3484	0.3472	0.3460	1	2	4	5	6	7	9	10	11
2.9	0.3448	0.3436	0.3425	0.3413	0.3401	0.3390	0.3378	0.3367	0.3356	0.3344	1	2	3	5	6	7	8	9	10
3.0	0.3333	0.3322	0.3311	0.3300	0.3289	0.3279	0.3268	0.3257	0.3247	0.3236	1	2	3	4	5	6	8	9	10
3.1	0.3226	0.3215	0.3205	0.3195	0.3185	0.3175	0.3165	0.3155	0.3145	0.3135	1	2	3	4	5	6	7	8	9
3.2	0.3125	0.3115	0.3106	0.3096	0.3086	0.3077	0.3067	0.3058	0.3049	0.3040	1	2	3	4	5	6	7	8	9
3.3	0.3030	0.3021	0.3012	0.3003	0.2994	0.2985	0.2976	0.2967	0.2959	0.2950	1	2	3	4	4	5	6	7	8
3.4	0.2941	0.2933	0.2924	0.2915	0.2907	0.2899	0.2890	0.2882	0.2874	0.2865	1	2	3	3	4	5	6	7	8
3.5	0.2857	0.2849	0.2841	0.2833	0.2825	0.2817	0.2809	0.2801	0.2793	0.2786	1	2	2	3	4	5	6	6	7
3.6	0.2778	0.2770	0.2762	0.2755	0.2747	0.2740	0.2732	0.2725	0.2717	0.2710	1	2	2	3	4	5	5	6	7
3.7	0.2703	0.2695	0.2688	0.2681	0.2674	0.2667	0.2660	0.2653	0.2646	0.2639	1	1	2	3	4	4	5	6	6
3.8	0.2632	0.2625	0.2618	0.2611	0.2604	0.2597	0.2591	0.2584	0.2577	0.2571	1	1	2	3	3	4	5	5	6
3.9	0.2564	0.2558	0.2551	0.2545	0.2538	0.2532	0.2525	0.2519	0.2513	0.2506	1	1	2	3	3	4	4	5	6
4.0	0.2500	0.2494	0.2488	0.2481	0.2475	0.2469	0.2463	0.2457	0.2451	0.2445	1	1	2	2	3	4	4	5	5
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

	$\frac{1}{x}$										Mean Differences (Subtract)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
4.1	0.2439	0.2433	0.2427	0.2421	0.2415	0.2410	0.2404	0.2398	0.2392	0.2387	1	1	2	2	3	3	4	5	5
4.2	0.2381	0.2375	0.2370	0.2364	0.2358	0.2353	0.2347	0.2342	0.2336	0.2331	1	1	2	2	3	3	4	4	5
4.3	0.2326	0.2320	0.2315	0.2309	0.2304	0.2299	0.2294	0.2288	0.2283	0.2278	1	1	2	2	3	3	4	4	5
4.4	0.2273	0.2268	0.2262	0.2257	0.2252	0.2247	0.2242	0.2237	0.2232	0.2227	1	1	2	2	3	3	4	4	5
4.5	0.2222	0.2217	0.2212	0.2208	0.2203	0.2198	0.2193	0.2188	0.2183	0.2179	0	1	1	2	2	3	3	4	4
4.6	0.2174	0.2169	0.2165	0.2160	0.2155	0.2151	0.2146	0.2141	0.2137	0.2132	0	1	1	2	2	3	3	4	4
4.7	0.2128	0.2123	0.2119	0.2114	0.2110	0.2105	0.2101	0.2096	0.2092	0.2088	0	1	1	2	2	3	3	4	4
4.8	0.2083	0.2079	0.2075	0.2070	0.2066	0.2062	0.2058	0.2053	0.2049	0.2045	0	1	1	2	2	3	3	3	4
4.9	0.2041	0.2037	0.2033	0.2028	0.2024	0.2020	0.2016	0.2012	0.2008	0.2004	0	1	1	2	2	2	3	3	4
5.0	0.2000	0.1996	0.1992	0.1988	0.1984	0.1980	0.1976	0.1972	0.1969	0.1965	0	1	1	2	2	2	3	3	4
5.1	0.1961	0.1957	0.1953	0.1949	0.1946	0.1942	0.1938	0.1934	0.1931	0.1927	0	1	1	2	2	2	3	3	3
5.2	0.1923	0.1919	0.1916	0.1912	0.1908	0.1905	0.1901	0.1898	0.1894	0.1890	0	1	1	1	2	2	3	3	3
5.3	0.1887	0.1883	0.1880	0.1876	0.1873	0.1869	0.1866	0.1862	0.1859	0.1855	0	1	1	1	2	2	2	3	3
5.4	0.1852	0.1848	0.1845	0.1842	0.1838	0.1835	0.1832	0.1828	0.1825	0.1821	0	1	1	1	2	2	2	3	3
5.5	0.1818	0.1815	0.1812	0.1808	0.1805	0.1802	0.1799	0.1795	0.1792	0.1789	0	1	1	1	2	2	2	3	3
5.6	0.1786	0.1783	0.1779	0.1776	0.1773	0.1770	0.1767	0.1764	0.1761	0.1757	0	1	1	1	2	2	2	3	3
5.7	0.1754	0.1751	0.1748	0.1745	0.1742	0.1739	0.1736	0.1733	0.1730	0.1727	0	1	1	1	2	2	2	2	3
5.8	0.1724	0.1721	0.1718	0.1715	0.1712	0.1709	0.1706	0.1704	0.1701	0.1698	0	1	1	1	1	2	2	2	3
5.9	0.1695	0.1692	0.1689	0.1686	0.1684	0.1681	0.1678	0.1675	0.1672	0.1669	0	1	1	1	1	2	2	2	3
6.0	0.1667	0.1664	0.1661	0.1658	0.1656	0.1653	0.1650	0.1647	0.1645	0.1642	0	1	1	1	1	2	2	2	2
6.1	0.1639	0.1637	0.1634	0.1631	0.1629	0.1626	0.1623	0.1621	0.1618	0.1616	0	1	1	1	1	2	2	2	2
6.2	0.1613	0.1610	0.1608	0.1605	0.1603	0.1600	0.1597	0.1595	0.1592	0.1590	0	1	1	1	1	2	2	2	2
6.3	0.1587	0.1585	0.1582	0.1580	0.1577	0.1575	0.1572	0.1570	0.1567	0.1565	0	0	1	1	1	1	2	2	2
6.4	0.1563	0.1560	0.1558	0.1555	0.1553	0.1550	0.1548	0.1546	0.1543	0.1541	0	0	1	1	1	1	2	2	2
6.5	0.1538	0.1536	0.1534	0.1531	0.1529	0.1527	0.1524	0.1522	0.1520	0.1517	0	0	1	1	1	1	2	2	2
6.6	0.1515	0.1513	0.1511	0.1508	0.1506	0.1504	0.1502	0.1499	0.1497	0.1495	0	0	1	1	1	1	2	2	2
6.7	0.1493	0.1490	0.1488	0.1486	0.1484	0.1481	0.1479	0.1477	0.1475	0.1473	0	0	1	1	1	1	2	2	2
6.8	0.1471	0.1468	0.1466	0.1464	0.1462	0.1460	0.1458	0.1456	0.1453	0.1451	0	0	1	1	1	1	1	2	2
6.9	0.1449	0.1447	0.1445	0.1443	0.1441	0.1439	0.1437	0.1435	0.1433	0.1431	0	0	1	1	1	1	1	2	2
7.0	0.1429	0.1427	0.1425	0.1422	0.1420	0.1418	0.1416	0.1414	0.1412	0.1410	0	0	1	1	1	1	1	2	2
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Math F2 - (BOOK) (DUMMY) Sept 4.indb 445

	$\frac{1}{x}$										Mean Differences (Subtract)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
7.1	0.1408	0.1406	0.1404	0.1403	0.1401	0.1399	0.1397	0.1395	0.1393	0.1391	0	0	1	1	1	1	1	2	2
7.2	0.1389	0.1387	0.1385	0.1383	0.1381	0.1379	0.1377	0.1376	0.1374	0.1372	0	0	1	1	1	1	1	2	2
7.3	0.1370	0.1368	0.1366	0.1364	0.1362	0.1361	0.1359	0.1357	0.1355	0.1353	0	0	1	1	1	1	1	1	2
7.4	0.1351	0.1350	0.1348	0.1346	0.1344	0.1342	0.1340	0.1339	0.1337	0.1335	0	0	1	1	1	1	1	1	2
7.5	0.1333	0.1332	0.1330	0.1328	0.1326	0.1325	0.1323	0.1321	0.1319	0.1318	0	0	1	1	1	1	1	1	2
7.6	0.1316	0.1314	0.1312	0.1311	0.1309	0.1307	0.1305	0.1304	0.1302	0.1300	0	0	1	1	1	1	1	1	2
7.7	0.1299	0.1297	0.1295	0.1294	0.1292	0.1290	0.1289	0.1287	0.1285	0.1284	0	0	0	1	1	1	1	1	1
7.8	0.1282	0.1280	0.1279	0.1277	0.1276	0.1274	0.1272	0.1271	0.1269	0.1267	0	0	0	1	1	1	1	1	1
7.9	0.1266	0.1264	0.1263	0.1261	0.1259	0.1258	0.1256	0.1255	0.1253	0.1252	0	0	0	1	1	1	1	1	1
8.0	0.1250	0.1248	0.1247	0.1245	0.1244	0.1242	0.1241	0.1239	0.1238	0.1236	0	0	0	1	1	1	1	1	1
8.1	0.1235	0.1233	0.1232	0.1230	0.1229	0.1227	0.1225	0.1224	0.1222	0.1221	0	0	0	1	1	1	1	1	1
8.2	0.1220	0.1218	0.1217	0.1215	0.1214	0.1212	0.1211	0.1209	0.1208	0.1206	0	0	0	1	1	1	1	1	1
8.3	0.1205	0.1203	0.1202	0.1200	0.1199	0.1198	0.1196	0.1195	0.1193	0.1192	0	0	0	1	1	1	1	1	1
8.4	0.1190	0.1189	0.1188	0.1186	0.1185	0.1183	0.1182	0.1181	0.1179	0.1178	0	0	0	1	1	1	1	1	1
8.5	0.1176	0.1175	0.1174	0.1172	0.1171	0.1170	0.1168	0.1167	0.1166	0.1164	0	0	0	1	1	1	1	1	1
8.6	0.1163	0.1161	0.1160	0.1159	0.1157	0.1156	0.1155	0.1153	0.1152	0.1151	0	0	0	1	1	1	1	1	1
8.7	0.1149	0.1148	0.1147	0.1145	0.1144	0.1143	0.1142	0.1140	0.1139	0.1138	0	0	0	1	1	1	1	1	1
8.8	0.1136	0.1135	0.1134	0.1133	0.1131	0.1130	0.1129	0.1127	0.1126	0.1125	0	0	0	1	1	1	1	1	1
8.9	0.1124	0.1122	0.1121	0.1120	0.1119	0.1117	0.1116	0.1115	0.1114	0.1112	0	0	0	0	1	1	1	1	1
9.0	0.1111	0.1110	0.1109	0.1107	0.1106	0.1105	0.1104	0.1103	0.1101	0.1100	0	0	0	0	1	1	1	1	1
9.1	0.1099	0.1098	0.1096	0.1095	0.1094	0.1093	0.1092	0.1091	0.1089	0.1088	0	0	0	0	1	1	1	1	1
9.2	0.1087	0.1086	0.1085	0.1083	0.1082	0.1081	0.1080	0.1079	0.1078	0.1076	0	0	0	0	1	1	1	1	1
9.3	0.1075	0.1074	0.1073	0.1072	0.1071	0.1070	0.1068	0.1067	0.1066	0.1065	0	0	0	0	1	1	1	1	1
9.4	0.1064	0.1063	0.1062	0.1060	0.1059	0.1058	0.1057	0.1056	0.1055	0.1054	0	0	0	0	1	1	1	1	1
9.5	0.1053	0.1052	0.1050	0.1049	0.1048	0.1047	0.1046	0.1045	0.1044	0.1043	0	0	0	0	1	1	1	1	1
9.6	0.1042	0.1041	0.1040	0.1038	0.1037	0.1036	0.1035	0.1034	0.1033	0.1032	0	0	0	0	1	1	1	1	1
9.7	0.1031	0.1030	0.1029	0.1028	0.1027	0.1026	0.1025	0.1024	0.1022	0.1021	0	0	0	0	1	1	1	1	1
9.8	0.1020	0.1019	0.1018	0.1017	0.1016	0.1015	0.1014	0.1013	0.1012	0.1011	0	0	0	0	1	1	1	1	1
9.9	0.1010	0.1009	0.1008	0.1007	0.1006	0.1005	0.1004	0.1003	0.1002	0.1001	0	0	0	0	1	1	1	1	1
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Cube Root

	$\sqrt[3]{x}$										Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.0000	1.0033	1.0066	1.0099	1.0132	1.0164	1.0196	1.0228	1.0260	1.0291	3	6	10	13	16	19	23	26	29
1.1	1.0323	1.0354	1.0385	1.0416	1.0446	1.0477	1.0507	1.0537	1.0567	1.0597	3	6	9	12	15	18	21	24	27
1.2	1.0627	1.0656	1.0685	1.0714	1.0743	1.0772	1.0801	1.0829	1.0858	1.0886	3	6	9	12	14	17	20	23	26
1.3	1.0914	1.0942	1.0970	1.0997	1.1025	1.1052	1.1079	1.1106	1.1133	1.1160	3	5	8	11	14	16	19	22	25
1.4	1.1187	1.1213	1.1240	1.1266	1.1292	1.1319	1.1344	1.1370	1.1396	1.1422	3	5	8	10	13	16	18	21	23
1.5	1.1447	1.1473	1.1498	1.1523	1.1548	1.1573	1.1598	1.1623	1.1647	1.1672	2	5	7	10	12	15	17	20	22
1.6	1.1696	1.1720	1.1745	1.1769	1.1793	1.1817	1.1840	1.1864	1.1888	1.1911	2	5	7	10	12	14	17	19	21
1.7	1.1935	1.1958	1.1981	1.2005	1.2028	1.2051	1.2074	1.2096	1.2119	1.2142	2	5	7	9	11	14	16	18	21
1.8	1.2164	1.2187	1.2209	1.2232	1.2254	1.2276	1.2298	1.2320	1.2342	1.2364	2	4	7	9	11	13	15	18	20
1.9	1.2386	1.2407	1.2429	1.2450	1.2472	1.2493	1.2515	1.2536	1.2557	1.2578	2	4	6	9	11	13	15	17	19
2.0	1.2599	1.2620	1.2641	1.2662	1.2683	1.2703	1.2724	1.2745	1.2765	1.2785	2	4	6	8	10	12	14	17	19
2.1	1.2806	1.2826	1.2846	1.2866	1.2887	1.2907	1.2927	1.2947	1.2966	1.2986	2	4	6	8	10	12	14	16	18
2.2	1.3006	1.3026	1.3045	1.3065	1.3084	1.3104	1.3123	1.3142	1.3162	1.3181	2	4	6	8	10	12	14	16	17
2.3	1.3200	1.3219	1.3238	1.3257	1.3276	1.3295	1.3314	1.3333	1.3351	1.3370	2	4	6	8	9	11	13	15	17
2.4	1.3389	1.3407	1.3426	1.3444	1.3463	1.3481	1.3499	1.3518	1.3536	1.3554	2	4	6	7	9	11	13	15	17
2.5	1.3572	1.3590	1.3608	1.3626	1.3644	1.3662	1.3680	1.3698	1.3715	1.3733	2	4	5	7	9	11	13	14	16
2.6	1.3751	1.3768	1.3786	1.3803	1.3821	1.3838	1.3856	1.3873	1.3890	1.3908	2	3	5	7	9	10	12	14	16
2.7	1.3925	1.3942	1.3959	1.3976	1.3993	1.4010	1.4027	1.4044	1.4061	1.4078	2	3	5	7	8	10	12	14	15
2.8	1.4095	1.4111	1.4128	1.4145	1.4161	1.4178	1.4195	1.4211	1.4228	1.4244	2	3	5	7	8	10	12	13	15
2.9	1.4260	1.4277	1.4293	1.4309	1.4326	1.4342	1.4358	1.4374	1.4390	1.4406	2	3	5	6	8	10	11	13	15
3.0	1.4422	1.4439	1.4454	1.4470	1.4486	1.4502	1.4518	1.4534	1.4550	1.4565	2	3	5	6	8	10	11	13	14
3.1	1.4581	1.4597	1.4612	1.4628	1.4643	1.4659	1.4674	1.4690	1.4705	1.4721	2	3	5	6	8	9	11	12	14
3.2	1.4736	1.4751	1.4767	1.4782	1.4797	1.4812	1.4828	1.4843	1.4858	1.4873	2	3	5	6	8	9	11	12	14
3.3	1.4888	1.4903	1.4918	1.4933	1.4948	1.4963	1.4978	1.4993	1.5007	1.5022	1	3	4	6	7	9	10	12	13
3.4	1.5037	1.5052	1.5066	1.5081	1.5096	1.5110	1.5125	1.5139	1.5154	1.5168	1	3	4	6	7	9	10	12	13
3.5	1.5183	1.5197	1.5212	1.5226	1.5241	1.5255	1.5269	1.5283	1.5298	1.5312	1	3	4	6	7	9	10	11	13
3.6	1.5326	1.5340	1.5355	1.5369	1.5383	1.5397	1.5411	1.5425	1.5439	1.5453	1	3	4	6	7	8	10	11	13
3.7	1.5467	1.5481	1.5495	1.5508	1.5522	1.5536	1.5550	1.5564	1.5577	1.5591	1	3	4	6	7	8	10	11	12
3.8	1.5605	1.5619	1.5632	1.5646	1.5659	1.5673	1.5687	1.5700	1.5714	1.5727	1	3	4	5	7	8	10	11	12
3.9	1.5741	1.5754	1.5767	1.5781	1.5794	1.5808	1.5821	1.5834	1.5848	1.5861	1	3	4	5	7	8	9	11	12
4.0	1.5874	1.5887	1.5900	1.5914	1.5927	1.5940	1.5953	1.5966	1.5979	1.5992	1	3	4	5	7	8	9	10	12
4.1	1.6005	1.6018	1.6031	1.6044	1.6057	1.6070	1.6083	1.6096	1.6109	1.6121	1	3	4	5	6	8	9	10	12
4.2	1.6134	1.6147	1.6160	1.6173	1.6185	1.6198	1.6211	1.6223	1.6236	1.6249	1	3	4	5	6	8	9	10	11
4.3	1.6261	1.6274	1.6287	1.6299	1.6312	1.6324	1.6337	1.6349	1.6362	1.6374	1	3	4	5	6	8	9	10	11
4.4	1.6386	1.6399	1.6411	1.6424	1.6436	1.6448	1.6461	1.6473	1.6485	1.6497	1	2	4	5	6	7	9	10	11
4.5	1.6510	1.6522	1.6534	1.6546	1.6558	1.6571	1.6583	1.6595	1.6607	1.6619	1	2	4	5	6	7	8	10	11
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

$\sqrt[3]{x}$											Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
4.6	1.6631	1.6643	1.6655	1.6667	1.6679	1.6691	1.6703	1.6715	1.6727	1.6739	1	2	4	5	6	7	8	10	11
4.7	1.6751	1.6763	1.6774	1.6786	1.6798	1.6810	1.6822	1.6833	1.6845	1.6857	1	2	4	5	6	7	8	9	11
4.8	1.6869	1.6880	1.6892	1.6904	1.6915	1.6927	1.6939	1.6950	1.6962	1.6973	1	2	3	5	6	7	8	9	10
4.9	1.6985	1.6997	1.7008	1.7020	1.7031	1.7043	1.7054	1.7065	1.7077	1.7088	1	2	3	5	6	7	8	9	10
5.0	1.7100	1.7111	1.7123	1.7134	1.7145	1.7157	1.7168	1.7179	1.7190	1.7202	1	2	3	5	6	7	8	9	10
5.1	1.7213	1.7224	1.7235	1.7247	1.7258	1.7269	1.7280	1.7291	1.7303	1.7314	1	2	3	4	6	7	8	9	10
5.2	1.7325	1.7336	1.7347	1.7358	1.7369	1.7380	1.7391	1.7402	1.7413	1.7424	1	2	3	4	6	7	8	9	10
5.3	1.7435	1.7446	1.7457	1.7468	1.7479	1.7490	1.7501	1.7512	1.7522	1.7533	1	2	3	4	5	7	8	9	10
5.4	1.7544	1.7555	1.7566	1.7577	1.7587	1.7598	1.7609	1.7620	1.7630	1.7641	1	2	3	4	5	6	8	9	10
5.5	1.7652	1.7662	1.7673	1.7684	1.7694	1.7705	1.7716	1.7726	1.7737	1.7748	1	2	3	4	5	6	7	9	10
5.6	1.7758	1.7769	1.7779	1.7790	1.7800	1.7811	1.7821	1.7832	1.7842	1.7853	1	2	3	4	5	6	7	8	9
5.7	1.7863	1.7874	1.7884	1.7894	1.7905	1.7915	1.7926	1.7936	1.7946	1.7957	1	2	3	4	5	6	7	8	9
5.8	1.7967	1.7977	1.7988	1.7998	1.8008	1.8018	1.8029	1.8039	1.8049	1.8059	1	2	3	4	5	6	7	8	9
5.9	1.8070	1.8080	1.8090	1.8100	1.8110	1.8121	1.8131	1.8141	1.8151	1.8161	1	2	3	4	5	6	7	8	9
6.0	1.8171	1.8181	1.8191	1.8201	1.8211	1.8222	1.8232	1.8242	1.8252	1.8262	1	2	3	4	5	6	7	8	9
6.1	1.8272	1.8282	1.8292	1.8302	1.8311	1.8321	1.8331	1.8341	1.8351	1.8361	1	2	3	4	5	6	7	8	9
6.2	1.8371	1.8381	1.8391	1.8400	1.8410	1.8420	1.8430	1.8440	1.8450	1.8459	1	2	3	4	5	6	7	8	9
6.3	1.8469	1.8479	1.8489	1.8498	1.8508	1.8518	1.8528	1.8537	1.8547	1.8557	1	2	3	4	5	6	7	8	9
6.4	1.8566	1.8576	1.8586	1.8595	1.8605	1.8615	1.8624	1.8634	1.8643	1.8653	1	2	3	4	5	6	7	8	9
6.5	1.8663	1.8672	1.8682	1.8691	1.8701	1.8710	1.8720	1.8729	1.8739	1.8748	1	2	3	4	5	6	7	8	9
6.6	1.8758	1.8767	1.8777	1.8786	1.8796	1.8805	1.8814	1.8824	1.8833	1.8843	1	2	3	4	5	6	7	8	8
6.7	1.8852	1.8861	1.8871	1.8880	1.8889	1.8899	1.8908	1.8917	1.8927	1.8936	1	2	3	4	5	6	7	7	8
6.8	1.8945	1.8955	1.8964	1.8973	1.8982	1.8992	1.9001	1.9010	1.9019	1.9029	1	2	3	4	5	6	6	7	8
6.9	1.9038	1.9047	1.9056	1.9065	1.9074	1.9084	1.9093	1.9102	1.9111	1.9120	1	2	3	4	5	5	6	7	8
7.0	1.9129	1.9138	1.9148	1.9157	1.9166	1.9175	1.9184	1.9193	1.9202	1.9211	1	2	3	4	5	5	6	7	8
7.1	1.9220	1.9229	1.9238	1.9247	1.9256	1.9265	1.9274	1.9283	1.9292	1.9301	1	2	3	4	4	5	6	7	8
7.2	1.9310	1.9319	1.9328	1.9337	1.9345	1.9354	1.9363	1.9372	1.9381	1.9390	1	2	3	4	4	5	6	7	8
7.3	1.9399	1.9408	1.9416	1.9425	1.9434	1.9443	1.9452	1.9461	1.9469	1.9478	1	2	3	4	4	5	6	7	8
7.4	1.9487	1.9496	1.9504	1.9513	1.9522	1.9531	1.9539	1.9548	1.9557	1.9566	1	2	3	3	4	5	6	7	8
7.5	1.9574	1.9583	1.9592	1.9600	1.9609	1.9618	1.9626	1.9635	1.9644	1.9652	1	2	3	3	4	5	6	7	8
7.6	1.9661	1.9670	1.9678	1.9687	1.9695	1.9704	1.9713	1.9721	1.9730	1.9738	1	2	3	3	4	5	6	7	8
7.7	1.9747	1.9755	1.9764	1.9772	1.9781	1.9789	1.9798	1.9806	1.9815	1.9823	1	2	3	3	4	5	6	7	8
7.8	1.9832	1.9840	1.9849	1.9857	1.9866	1.9874	1.9883	1.9891	1.9899	1.9908	1	2	3	3	4	5	6	7	8
7.9	1.9916	1.9925	1.9933	1.9941	1.9950	1.9958	1.9967	1.9975	1.9983	1.9992	1	2	3	3	4	5	6	7	8
8.0	2.0000	2.0008	2.0017	2.0025	2.0033	2.0042	2.0050	2.0058	2.0066	2.0075	1	2	2	3	4	5	6	7	7
8.1	2.0083	2.0091	2.0100	2.0108	2.0116	2.0124	2.0132	2.0141	2.0149	2.0157	1	2	2	3	4	5	6	7	7
8.2	2.0165	2.0173	2.0182	2.0190	2.0198	2.0206	2.0214	2.0223	2.0231	2.0239	1	2	2	3	4	5	6	7	7
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

$\sqrt[3]{x}$											Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
8.3	2.0247	2.0255	2.0263	2.0271	2.0279	2.0288	2.0296	2.0304	2.0312	2.0320	1	2	2	3	4	5	6	6	7
8.4	2.0328	2.0336	2.0344	2.0352	2.0360	2.0368	2.0376	2.0384	2.0392	2.0400	1	2	2	3	4	5	6	6	7
8.5	2.0408	2.0416	2.0424	2.0432	2.0440	2.0448	2.0456	2.0464	2.0472	2.0480	1	2	2	3	4	5	6	6	7
8.6	2.0488	2.0496	2.0504	2.0512	2.0520	2.0528	2.0536	2.0543	2.0551	2.0559	1	2	2	3	4	5	6	6	7
8.7	2.0567	2.0575	2.0583	2.0591	2.0599	2.0606	2.0614	2.0622	2.0630	2.0638	1	2	2	3	4	5	5	6	7
8.8	2.0646	2.0653	2.0661	2.0669	2.0677	2.0685	2.0692	2.0700	2.0708	2.0716	1	2	2	3	4	5	5	6	7
8.9	2.0724	2.0731	2.0739	2.0747	2.0755	2.0762	2.0770	2.0778	2.0785	2.0793	1	2	2	3	4	5	5	6	7
9.0	2.0801	2.0809	2.0816	2.0824	2.0832	2.0839	2.0847	2.0855	2.0862	2.0870	1	2	2	3	4	5	5	6	7
9.1	2.0878	2.0885	2.0893	2.0901	2.0908	2.0916	2.0923	2.0931	2.0939	2.0946	1	2	2	3	4	5	5	6	7
9.2	2.0954	2.0961	2.0969	2.0977	2.0984	2.0992	2.0999	2.1007	2.1014	2.1022	1	2	2	3	4	5	5	6	7
9.3	2.1029	2.1037	2.1045	2.1052	2.1060	2.1067	2.1075	2.1082	2.1090	2.1097	1	2	2	3	4	5	5	6	7
9.4	2.1105	2.1112	2.1120	2.1127	2.1134	2.1142	2.1149	2.1157	2.1164	2.1172	1	1	2	3	4	4	5	6	7
9.5	2.1179	2.1187	2.1194	2.1201	2.1209	2.1216	2.1224	2.1231	2.1238	2.1246	1	1	2	3	4	4	5	6	7
9.6	2.1253	2.1261	2.1268	2.1275	2.1283	2.1290	2.1297	2.1305	2.1312	2.1319	1	1	2	3	4	4	5	6	7
9.7	2.1327	2.1334	2.1341	2.1349	2.1356	2.1363	2.1371	2.1378	2.1385	2.1392	1	1	2	3	4	4	5	6	7
9.8	2.1400	2.1407	2.1414	2.1422	2.1429	2.1436	2.1443	2.1451	2.1458	2.1465	1	1	2	3	4	4	5	6	7
9.9	2.1472	2.1480	2.1487	2.1494	2.1501	2.1508	2.1516	2.1523	2.1530	2.1537	1	1	2	3	4	4	5	6	6
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Cube Root

	$\sqrt[3]{x}$										Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	2.1544	2.1616	2.1687	2.1758	2.1828	2.1898	2.1967	2.2036	2.2104	2.2172	7	14	21	28	35	42	49	56	63
11	2.2240	2.2307	2.2374	2.2440	2.2506	2.2572	2.2637	2.2702	2.2766	2.2831	7	13	20	26	33	39	46	52	59
12	2.2894	2.2958	2.3021	2.3084	2.3146	2.3208	2.3270	2.3331	2.3392	2.3453	6	12	19	25	31	37	43	50	56
13	2.3513	2.3573	2.3633	2.3693	2.3752	2.3811	2.3870	2.3928	2.3986	2.4044	6	12	18	24	29	35	41	47	53
14	2.4101	2.4159	2.4216	2.4272	2.4329	2.4385	2.4441	2.4497	2.4552	2.4607	6	11	17	22	28	34	39	45	50
15	2.4662	2.4717	2.4771	2.4825	2.4879	2.4933	2.4987	2.5040	2.5093	2.5146	5	11	16	21	27	32	38	43	48
16	2.5198	2.5251	2.5303	2.5355	2.5407	2.5458	2.5510	2.5561	2.5612	2.5662	5	10	15	21	26	31	36	41	46
17	2.5713	2.5763	2.5813	2.5863	2.5913	2.5962	2.6012	2.6061	2.6110	2.6159	5	10	15	20	25	30	35	40	45
18	2.6207	2.6256	2.6304	2.6352	2.6400	2.6448	2.6495	2.6543	2.6590	2.6637	5	10	14	19	24	29	33	38	43
19	2.6684	2.6731	2.6777	2.6824	2.6870	2.6916	2.6962	2.7008	2.7053	2.7099	5	9	14	18	23	28	32	37	41
20	2.7144	2.7189	2.7234	2.7279	2.7324	2.7369	2.7413	2.7457	2.7501	2.7545	4	9	13	18	22	27	31	36	40
21	2.7589	2.7633	2.7677	2.7720	2.7763	2.7806	2.7850	2.7892	2.7935	2.7978	4	9	13	17	22	26	30	35	39
22	2.8020	2.8063	2.8105	2.8147	2.8189	2.8231	2.8273	2.8314	2.8356	2.8397	4	8	13	17	21	25	29	33	38
23	2.8439	2.8480	2.8521	2.8562	2.8603	2.8643	2.8684	2.8724	2.8765	2.8805	4	8	12	16	20	24	28	33	37
24	2.8845	2.8885	2.8925	2.8965	2.9004	2.9044	2.9083	2.9123	2.9162	2.9201	4	8	12	16	20	24	28	32	36
25	2.9240	2.9279	2.9318	2.9357	2.9395	2.9434	2.9472	2.9511	2.9549	2.9587	4	8	12	15	19	23	27	31	35
26	2.9625	2.9663	2.9701	2.9738	2.9776	2.9814	2.9851	2.9888	2.9926	2.9963	4	8	11	15	19	23	26	30	34
27	3.0000	3.0037	3.0074	3.0111	3.0147	3.0184	3.0221	3.0257	3.0293	3.0330	4	7	11	15	18	22	26	29	33
28	3.0366	3.0402	3.0438	3.0474	3.0510	3.0546	3.0581	3.0617	3.0652	3.0688	4	7	11	14	18	21	25	29	32
29	3.0723	3.0758	3.0794	3.0829	3.0864	3.0899	3.0934	3.0968	3.1003	3.1038	3	7	10	14	17	21	24	28	31
30	3.1072	3.1107	3.1141	3.1176	3.1210	3.1244	3.1278	3.1312	3.1346	3.1380	3	7	10	14	17	20	24	27	31
31	3.1414	3.1448	3.1481	3.1515	3.1548	3.1582	3.1615	3.1648	3.1682	3.1715	3	7	10	13	17	20	23	27	30
32	3.1748	3.1781	3.1814	3.1847	3.1880	3.1913	3.1945	3.1978	3.2010	3.2043	3	7	10	13	16	20	23	26	29
33	3.2075	3.2108	3.2140	3.2172	3.2204	3.2237	3.2269	3.2301	3.2332	3.2364	3	6	10	13	16	19	22	26	29
34	3.2396	3.2428	3.2460	3.2491	3.2523	3.2554	3.2586	3.2617	3.2648	3.2679	3	6	9	13	16	19	22	25	28
35	3.2711	3.2742	3.2773	3.2804	3.2835	3.2866	3.2897	3.2927	3.2958	3.2989	3	6	9	12	15	19	22	25	28
36	3.3019	3.3050	3.3080	3.3111	3.3141	3.3171	3.3202	3.3232	3.3262	3.3292	3	6	9	12	15	18	21	24	27
37	3.3322	3.3352	3.3382	3.3412	3.3442	3.3472	3.3501	3.3531	3.3561	3.3590	3	6	9	12	15	18	21	24	27
38	3.3620	3.3649	3.3679	3.3708	3.3737	3.3767	3.3796	3.3825	3.3854	3.3883	3	6	9	12	15	18	20	23	26
39	3.3912	3.3941	3.3970	3.3999	3.4028	3.4056	3.4085	3.4114	3.4142	3.4171	3	6	9	12	14	17	20	23	26
40	3.4200	3.4228	3.4256	3.4285	3.4313	3.4341	3.4370	3.4398	3.4426	3.4454	3	6	8	11	14	17	20	23	25
41	3.4482	3.4510	3.4538	3.4566	3.4594	3.4622	3.4650	3.4677	3.4705	3.4733	3	6	8	11	14	17	19	22	25
42	3.4760	3.4788	3.4815	3.4843	3.4870	3.4898	3.4925	3.4952	3.4980	3.5007	3	5	8	11	14	16	19	22	25
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

$\sqrt{x}$											Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
43	3.5034	3.5061	3.5088	3.5115	3.5142	3.5169	3.5196	3.5223	3.5250	3.5277	3	5	8	11	13	16	19	22	24
44	3.5303	3.5330	3.5357	3.5384	3.5410	3.5437	3.5463	3.5490	3.5516	3.5543	3	5	8	11	13	16	19	21	24
45	3.5569	3.5595	3.5622	3.5648	3.5674	3.5700	3.5726	3.5752	3.5778	3.5804	3	5	8	10	13	16	18	21	24
46	3.5830	3.5856	3.5882	3.5908	3.5934	3.5960	3.5986	3.6011	3.6037	3.6063	3	5	8	10	13	15	18	21	23
47	3.6088	3.6114	3.6139	3.6165	3.6190	3.6216	3.6241	3.6267	3.6292	3.6317	3	5	8	10	13	15	18	20	23
48	3.6342	3.6368	3.6393	3.6418	3.6443	3.6468	3.6493	3.6518	3.6543	3.6568	3	5	8	10	13	15	18	20	23
49	3.6593	3.6618	3.6643	3.6668	3.6692	3.6717	3.6742	3.6766	3.6791	3.6816	2	5	7	10	12	15	17	20	22
50	3.6840	3.6865	3.6889	3.6914	3.6938	3.6963	3.6987	3.7011	3.7036	3.7060	2	5	7	10	12	15	17	20	22
51	3.7084	3.7109	3.7133	3.7157	3.7181	3.7205	3.7229	3.7253	3.7277	3.7301	2	5	7	10	12	14	17	19	22
52	3.7325	3.7349	3.7373	3.7397	3.7421	3.7444	3.7468	3.7492	3.7516	3.7539	2	5	7	10	12	14	17	19	21
53	3.7563	3.7586	3.7610	3.7634	3.7657	3.7681	3.7704	3.7728	3.7751	3.7774	2	5	7	9	12	14	16	19	21
54	3.7798	3.7821	3.7844	3.7867	3.7891	3.7914	3.7937	3.7960	3.7983	3.8006	2	5	7	9	12	14	16	19	21
55	3.8030	3.8053	3.8076	3.8099	3.8121	3.8144	3.8167	3.8190	3.8213	3.8236	2	5	7	9	11	14	16	18	21
56	3.8259	3.8281	3.8304	3.8327	3.8349	3.8372	3.8395	3.8417	3.8440	3.8462	2	5	7	9	11	14	16	18	20
57	3.8485	3.8508	3.8530	3.8552	3.8575	3.8597	3.8620	3.8642	3.8664	3.8687	2	4	7	9	11	13	16	18	20
58	3.8709	3.8731	3.8753	3.8775	3.8798	3.8820	3.8842	3.8864	3.8886	3.8908	2	4	7	9	11	13	15	18	20
59	3.8930	3.8952	3.8974	3.8996	3.9018	3.9040	3.9061	3.9083	3.9105	3.9127	2	4	7	9	11	13	15	17	20
60	3.9149	3.9170	3.9192	3.9214	3.9235	3.9257	3.9279	3.9300	3.9322	3.9343	2	4	6	9	11	13	15	17	19
61	3.9365	3.9386	3.9408	3.9429	3.9451	3.9472	3.9494	3.9515	3.9536	3.9558	2	4	6	9	11	13	15	17	19
62	3.9579	3.9600	3.9621	3.9643	3.9664	3.9685	3.9706	3.9727	3.9748	3.9770	2	4	6	8	11	13	15	17	19
63	3.9791	3.9812	3.9833	3.9854	3.9875	3.9896	3.9916	3.9937	3.9958	3.9979	2	4	6	8	10	13	15	17	19
64	4.0000	4.0021	4.0042	4.0062	4.0083	4.0104	4.0125	4.0145	4.0166	4.0187	2	4	6	8	10	12	15	17	19
65	4.0207	4.0228	4.0248	4.0269	4.0290	4.0310	4.0331	4.0351	4.0372	4.0392	2	4	6	8	10	12	14	16	18
66	4.0412	4.0433	4.0453	4.0474	4.0494	4.0514	4.0534	4.0555	4.0575	4.0595	2	4	6	8	10	12	14	16	18
67	4.0615	4.0636	4.0656	4.0676	4.0696	4.0716	4.0736	4.0756	4.0776	4.0797	2	4	6	8	10	12	14	16	18
68	4.0817	4.0837	4.0857	4.0876	4.0896	4.0916	4.0936	4.0956	4.0976	4.0996	2	4	6	8	10	12	14	16	18
69	4.1016	4.1035	4.1055	4.1075	4.1095	4.1114	4.1134	4.1154	4.1174	4.1193	2	4	6	8	10	12	14	16	18
70	4.1213	4.1232	4.1252	4.1272	4.1291	4.1311	4.1330	4.1350	4.1369	4.1389	2	4	6	8	10	12	14	16	18
71	4.1408	4.1428	4.1447	4.1466	4.1486	4.1505	4.1524	4.1544	4.1563	4.1582	2	4	6	8	10	12	14	15	17
72	4.1602	4.1621	4.1640	4.1659	4.1679	4.1698	4.1717	4.1736	4.1755	4.1774	2	4	6	8	10	12	13	15	17
73	4.1793	4.1812	4.1832	4.1851	4.1870	4.1889	4.1908	4.1927	4.1946	4.1964	2	4	6	8	10	11	13	15	17
74	4.1983	4.2002	4.2021	4.2040	4.2059	4.2078	4.2097	4.2115	4.2134	4.2153	2	4	6	8	9	11	13	15	17
75	4.2172	4.2190	4.2209	4.2228	4.2246	4.2265	4.2284	4.2302	4.2321	4.2340	2	4	6	7	9	11	13	15	17
76	4.2358	4.2377	4.2395	4.2414	4.2432	4.2451	4.2469	4.2488	4.2506	4.2525	2	4	6	7	9	11	13	15	17
77	4.2543	4.2562	4.2580	4.2598	4.2617	4.2635	4.2653	4.2672	4.2690	4.2708	2	4	6	7	9	11	13	15	17
78	4.2727	4.2745	4.2763	4.2781	4.2799	4.2818	4.2836	4.2854	4.2872	4.2890	2	4	5	7	9	11	13	15	16
79	4.2908	4.2927	4.2945	4.2963	4.2981	4.2999	4.3017	4.3035	4.3053	4.3071	2	4	5	7	9	11	13	14	16
80	4.3089	4.3107	4.3125	4.3142	4.3160	4.3178	4.3196	4.3214	4.3232	4.3250	2	4	5	7	9	11	13	14	16
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

	$\sqrt[3]{x}$										Mean Differences (Add)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
81	4.3267	4.3285	4.3303	4.3321	4.3339	4.3356	4.3374	4.3392	4.3409	4.3427	2	4	5	7	9	11	12	14	16
82	4.3445	4.3462	4.3480	4.3498	4.3515	4.3533	4.3551	4.3568	4.3586	4.3603	2	4	5	7	9	11	12	14	16
83	4.3621	4.3638	4.3656	4.3673	4.3691	4.3708	4.3726	4.3743	4.3760	4.3778	2	3	5	7	9	10	12	14	16
84	4.3795	4.3813	4.3830	4.3847	4.3865	4.3882	4.3899	4.3917	4.3934	4.3951	2	3	5	7	9	10	12	14	16
85	4.3968	4.3986	4.4003	4.4020	4.4037	4.4054	4.4072	4.4089	4.4106	4.4123	2	3	5	7	9	10	12	14	15
86	4.4140	4.4157	4.4174	4.4191	4.4208	4.4225	4.4242	4.4259	4.4276	4.4293	2	3	5	7	9	10	12	14	15
87	4.4310	4.4327	4.4344	4.4361	4.4378	4.4395	4.4412	4.4429	4.4446	4.4463	2	3	5	7	8	10	12	14	15
88	4.4480	4.4496	4.4513	4.4530	4.4547	4.4564	4.4580	4.4597	4.4614	4.4631	2	3	5	7	8	10	12	13	15
89	4.4647	4.4664	4.4681	4.4698	4.4714	4.4731	4.4748	4.4764	4.4781	4.4797	2	3	5	7	8	10	12	13	15
90	4.4814	4.4831	4.4847	4.4864	4.4880	4.4897	4.4913	4.4930	4.4946	4.4963	2	3	5	7	8	10	12	13	15
91	4.4979	4.4996	4.5012	4.5029	4.5045	4.5062	4.5078	4.5094	4.5111	4.5127	2	3	5	7	8	10	11	13	15
92	4.5144	4.5160	4.5176	4.5193	4.5209	4.5225	4.5241	4.5258	4.5274	4.5290	2	3	5	7	8	10	11	13	15
93	4.5307	4.5323	4.5339	4.5355	4.5371	4.5388	4.5404	4.5420	4.5436	4.5452	2	3	5	6	8	10	11	13	15
94	4.5468	4.5484	4.5501	4.5517	4.5533	4.5549	4.5565	4.5581	4.5597	4.5613	2	3	5	6	8	10	11	13	14
95	4.5629	4.5645	4.5661	4.5677	4.5693	4.5709	4.5725	4.5741	4.5757	4.5773	2	3	5	6	8	10	11	13	14
96	4.5789	4.5804	4.5820	4.5836	4.5852	4.5868	4.5884	4.5900	4.5915	4.5931	2	3	5	6	8	10	11	13	14
97	4.5947	4.5963	4.5979	4.5994	4.6010	4.6026	4.6042	4.6057	4.6073	4.6089	2	3	5	6	8	9	11	13	14
98	4.6104	4.6120	4.6136	4.6151	4.6167	4.6183	4.6198	4.6214	4.6229	4.6245	2	3	5	6	8	9	11	13	14
99	4.6261	4.6276	4.6292	4.6307	4.6323	4.6338	4.6354	4.6369	4.6385	4.6400	2	3	5	6	8	9	11	12	14
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Glossary

Algebra	A branch of mathematics dealing with manipulation of symbols, numbers and basic arithmetic operations
Angle	Measure of how wide apart two intersecting straight lines are at their point of intersection
Angle of depression	Angle between a horizontal line and a line intersecting the horizontal line from below
Angle of elevation	Angle between a horizontal line and a line intersecting the horizontal line from above
Arithmetic	A branch of mathematics that is concerned with the operations of addition, subtraction, multiplication, and division of numbers
Bar chart	A graph that presents data using rectangular bars of heights or lengths proportional to the values they represent
Binary operation	An operation that applies to two quantities or expressions. Usually, symbols are used to stand for one or more arithmetic operations
Centre of rotation	A point about which a plane figure rotates. This point does not move during the rotation
Characteristic	The integral part of a logarithm. This is the exponent of 10 when a number is expressed in the scientific form
Coefficient of a term	A constant multiple of a variable or variables
Class boundaries	The actual class limits of a class interval
Class interval	The range of values from the lower to the upper class limits
Class limits	The first and last numbers of each class interval. The minimum number is called the lower class limit and the maximum number is called the upper class limit

Common logarithms	Logarithms of numbers to base 10
Combined transformation	Two or more transformations carried out, one after the other, to obtain a final transformation of the original object
Completing the square	A method used to solve a quadratic equation by changing the form of the equation
Congruent figures	Plane figures having equal side lengths and equal angles
Complement of a set	A set containing all elements of a universal set which are not members of a particular set from the universal set
Corresponding sides	Sides of different triangles or shapes that are in the same position
Cosine of an angle	Ratio between the adjacent side of an angle and hypotenus of a right – angle angled triangle
Cube root of a number	A number, which when multiplied by itself three times gives the given number
Cumulative frequency	A progressive sum of the frequencies of a frequency distribution
Data	Pieces of information, usually numeric that are collected through observation or calculation
Difference of two squares	An expression of the form of $a^2 - b^2$
Empty set	A set which has no elements
Enlargement	A transformation in which a figure is made larger
Equations	A mathematical statement consisting of the equal symbol (=) between two algebraic expressions
Equiangular polygon	A polygon whose vertex angles are equal. For example a square, an equilateral triangle, and rectangle
Exponent	The number of times a number is multiplied by itself

Expressions	A mathematical phrase that combines numbers and/or variables using mathematical operations
Factorisation of an algebraic expression	Process of splitting an algebraic expression into its factors
Finite set	A set with a known number of elements
Splitting the middle term of a quadratic expression	Expressing the middle term as a sum of two terms to satisfy certain conditions
Frequency	The number of times an event occurs in a given set of data
Frequency polygon	A line graph of class frequency plotted against class midpoint. It can be obtained by joining the midpoints of the tops of the rectangles in the histogram
Hypotenuse	The longest side of a right-angled triangle. It is also opposite to the right angle.
Identities	Equations which are true for all values of the variable. This means that if any value of the unknown is substituted on both sides, each side gives the same number
Infinite set	A set whose number of elements cannot be counted
Improper subset	A subset containing all elements of the original set
Isosceles triangle	A triangle with two sides of equal length
Line of symmetry	An imaginary line passing through the centre of an object which divides the object in two halves that are mirror images of each other
Linear expression	An algebraic expression whose highest exponent of the variable is 1
Linear factors	A factor of an algebraic expression which is linear
Logarithm of a number	The exponent of a number expressed in exponential form
Lower class boundary	A boundary found by subtracting 0.5 units from the lower class limit
Mantissa	The part of a logarithm which is located after a decimal point. Mantissa is always positive

Members of a set	Any object that belongs to a set
Ogive	A smooth curve that plots cumulative frequency on the y – axis and upper class boundaries along the x – axis
Perfect square	A quadratic expression whose factors are identical
Pictograms	Pictures or symbols used for presenting statistical information
Pie chart	A circular graph that presents statistic information
Polygon	A plane shape made up of straight line segments, that are connected to each other end to end to form a closed figure
Postulates	Statements of facts which are assumed to be true
Prime factorisation	Expressing a number as a product of prime numbers
Proper subset	A subset of a set which is not equal to the set
Proof	An argument which systematically shows that the stated assumptions logically lead to the conclusion
Proportion	An expression stating that one quantity is a constant multiple of another quantity
Pythagoras' theorem	The square of the hypotenuse of a right – angled triangle is equal to the sum of the squares of the other two sides
Quadratic expression	An expression of the form $ax^2 + bx + c$ where a, b, and c are real numbers, $a \neq 0$
Quadrilateral	A four sided polygon having four sides and four vertices
Radical	A symbol that represents a particular root of a number. $\sqrt[n]{A}$ denotes n – th roof of A.
Ratio of lengths	The value obtained by comparing the lengths of sides of similar triangles or shapes
Rationalising a denominator	Elimination of radical expressions in the denominator such as square roots and cube roots

Reciprocal	Inverse of a number, usually obtained by interchanging the numerator and denominator
Reflection	A transformation which reflects all points of a plane in a line called line of symmetry
n^{th} root of a number	A number which when multiplied by itself n times gives the original number
Rotation	A transformation which turns an object through a certain angle. Description of a rotation considers the centre of rotation, angle of rotation and direction of rotation
Scale factor	The ratio between corresponding measurements of the original object and a new object after enlargement. If the scale factor is greater than 1, the new figure will be larger. If the scale factor is a fraction, the new figure will be smaller. A scale factor is also called enlargement factor
Scientific notation of writing numbers	A way of writing a number in the form of $A \times 10^n$ where n is an integer and $1 \leq A < 10$. It is also called standard form of a number
Set	Collection of things which have common properties
Similar figures:	Geometrical figures whose corresponding angles are equal and corresponding sides are proportional. Similar figures have the same shape
Sine of an angle	The value obtained by dividing the length of the opposite side of a right – angled triangle by the length of the hypotenuse
Solving quadratic equation	Process of finding the two roots of the equation
Special angles	Angle $\theta°$ where $\theta° = 30°, 45°, 60°$ whose trigonometric ratios are easily found
Square root	A number which when multiplied by itself, gives the given number
Square	A plane figure with four equal sides and four right angles

Statistics	Branch of mathematics which deals with data collection, organisation, analysis and interpretation
Subset	A set which is contained in another set
Tangent of an angle	Ratio between the opposite and adjacent sides of an angle of a right – angled triangle.
Theorem	A statement of assumptions and the conclusion that follows from them
Transformation	A change of shape, position, size or direction of objects. The basic types of transformations are reflection, rotation, translation and enlargement
Translation	Transformation of an object by moving it along a straight line
Trigonometric ratios	Sine, cosine and tangent of an angle of a right – angled triangle
Universal set	A set containing all elements of all other sets under consideration
Upper class boundary	A boundary found by adding 0.5 units to the upper class limit
Venn diagrams	A geometrical illustration that uses circles to present sets and set operations
Word problem	A literal description of the mathematical formulation of a real life problem.
$x-y$ plane	A plane that contains two perpendicular coordinates axes x and y. The x – axis is placed horizontally and y – axis is placed vertically and they meet at a point called the origin

Bibliography

Macrae M. & Nessoro S. (2008). New General Mathematics Form 2 Students' Book, Edinburgh Gate: Pearson Longman.

Ministry of Education and Vocational Training, (2005). Basic Mathematics Syllabus for Ordinary Secondary Education Form I – IV, Dar es Salaam.

Ministry of Education and Vocational Training, (2009). Secondary Basic Mathematics, Educational Books Dar es Salaam: Publishers Ltd.

SCSU & MoEVT. Zanzibar. Rev. Ed. (2010). Mathematics for Secondary Schools Form Two Students Book, Dar es Salaam: Uhuru Media Ltd.

Institute of Curriculum development, (1991). Secondary Basic Mathematics Book Two, Dar Es Salaam.

NECTA (2016) Mathematical tables and formulae for Ordinary and Advanced levels, Secondary Education, Dar es Salaam.

Plews, A. M (1979). Introductory Statistics. Heinemann Educational: Oxford.

Streeter, J. et.al (1989). Intermediate Algebra Third Edition. Mac Graw Hill, Boston.

Index

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